# Contents

### Preface to the Second Edition, vii

Preface to the First Edition, ix

### **Preliminaries**, 1

1

Part 1 Preliminaries, 1 Part 2 Algebraic Structures, 16

### Part I—Basic Linear Algebra, 31

Vector Spaces, 33 Vector Spaces, 33 Subspaces, 35 Direct Sums, 38 Spanning Sets and Linear Independence, 41 The Dimension of a Vector Space, 44 Ordered Bases and Coordinate Matrices, 47 The Row and Column Spaces of a Matrix, 48 The Complexification of a Real Vector Space, 49 Exercises, 51

# 2 Linear Transformations, 55

Linear Transformations, 55 Isomorphisms, 57 The Kernel and Image of a Linear Transformation, 57 Linear Transformations from  $F^n$  to  $F^m$ , 59 The Rank Plus Nullity Theorem, 59 Change of Basis Matrices, 60 The Matrix of a Linear Transformation, 61 Change of Bases for Linear Transformations, 63 Equivalence of Matrices, 64 Similarity of Matrices, 65 Similarity of Operators, 66 Invariant Subspaces and Reducing Pairs, 68 3

Topological Vector Spaces, 68 Linear Operators on  $V^{\mathbb{C}}$ , 71 Exercises, 72 **The Isomorphism Theorems, 75** Quotient Spaces, 75 The Universal Property of Quotients and the First Isomorphism Theorem, 77 Quotient Spaces, Complements and Codimension, 79

Additional Isomorphism Theorems, 80 Linear Functionals, 82 Dual Bases, 83 Reflexivity, 84 Annihilators, 86 Operator Adjoints, 88 Exercises, 90

# 4 Modules I: Basic Properties, 93

Modules, 93 Motivation, 93 Submodules, 95 Spanning Sets, 96 Linear Independence, 98 Torsion Elements, 99 Annihilators, 99 Free Modules, 99 Homomorphisms, 100 Quotient Modules, 101 The Correspondence and Isomorphism Theorems, 102 Direct Sums and Direct Summands, 102 Modules Are Not As Nice As Vector Spaces, 106 Exercises, 106

# 5 Modules II: Free and Noetherian Modules, 109 The Rank of a Free Module, 109 Free Modules and Epimorphisms, 114 Noetherian Modules, 115 The Hilbert Basis Theorem, 118 Exercises, 119

Modules over a Principal Ideal Domain, 121
 Annihilators and Orders, 121
 Cyclic Modules, 122
 Free Modules over a Principal Ideal Domain, 123
 Torsion-Free and Free Modules, 125

Prelude to Decomposition: Cyclic Modules, 126 The First Decomposition, 127 A Look Ahead, 127 The Primary Decomposition, 128 The Cyclic Decomposition of a Primary Module, 130 The Primary Cyclic Decomposition Theorem, 134 The Invariant Factor Decomposition, 135 Exercises, 138 The Structure of a Linear Operator, 141 A Brief Review, 141 The Module Associated with a Linear Operator, 142 Orders and the Minimal Polynomial, 144 Cyclic Submodules and Cyclic Subspaces, 145 Summary, 147 The Decomposition of  $V_{\tau}$ , 147 The Rational Canonical Form, 148 Exercises, 151 **Eigenvalues and Eigenvectors**, 153 The Characteristic Polynomial of an Operator, 153 Eigenvalues and Eigenvectors, 155 Geometric and Algebraic Multiplicities, 157 The Jordan Canonical Form, 158 Triangularizability and Schur's Lemma, 160 Diagonalizable Operators, 165 Projections, 166 The Algebra of Projections, 167 Resolutions of the Identity, 170 Spectral Resolutions, 172 Projections and Invariance, 173 Exercises, 174 **Real and Complex Inner Product Spaces**, 181 Norm and Distance, 183 Isometries, 186 Orthogonality, 187 Orthogonal and Orthonormal Sets, 188 The Projection Theorem and Best Approximations, 192 Orthogonal Direct Sums, 194 The Riesz Representation Theorem, 195 Exercises, 196

7

8

9

#### 10 Structure Theory for Normal Operators, 201 The Adjoint of a Lincor Operator 201

The Adjoint of a Linear Operator, 201

Unitary Diagonalizability, 204 Normal Operators, 205 Special Types of Normal Operators, 207 Self-Adjoint Operators, 208 Unitary Operators and Isometries, 210 The Structure of Normal Operators, 215 Matrix Versions, 222 Orthogonal Projections, 223 Orthogonal Resolutions of the Identity, 226 The Spectral Theorem, 227 Spectral Resolutions and Functional Calculus, 228 Positive Operators, 230 The Polar Decomposition of an Operator, 232 Exercises, 234

# Part II—Topics, 235

Metric Vector Spaces: The Theory of Bilinear Forms, 239 11 Symmetric, Skew-Symmetric and Alternate Forms, 239 The Matrix of a Bilinear Form, 242 Quadratic Forms, 244 Orthogonality, 245 Linear Functionals, 248 Orthogonal Complements and Orthogonal Direct Sums, 249 Isometries, 252 Hyperbolic Spaces, 253 Nonsingular Completions of a Subspace, 254 The Witt Theorems: A Preview, 256 The Classification Problem for Metric Vector Spaces, 257 Symplectic Geometry, 258 The Structure of Orthogonal Geometries: Orthogonal Bases, 264 The Classification of Orthogonal Geometries: Canonical Forms, 266 The Orthogonal Group, 272 The Witt's Theorems for Orthogonal Geometries, 275 Maximal Hyperbolic Subspaces of an Orthogonal Geometry, 277 Exercises, 279

# 12 Metric Spaces, 283

The Definition, 283 Open and Closed Sets, 286 Convergence in a Metric Space, 287 The Closure of a Set, 288

### Contents xv

Dense Subsets, 290 Continuity, 292 Completeness, 293 Isometries, 297 The Completion of a Metric Space, 298 Exercises, 303

# 13 Hilbert Spaces, 307

A Brief Review, 307 Hilbert Spaces, 308 Infinite Series, 312 An Approximation Problem, 313 Hilbert Bases, 317 Fourier Expansions, 318 A Characterization of Hilbert Bases, 328 Hilbert Dimension, 328 A Characterization of Hilbert Spaces, 329 The Riesz Representation Theorem, 331 Exercises, 334

### 14 Tensor Products, 337

Universality, 337 Bilinear Maps, 341 Tensor Products, 343 When Is a Tensor Product Zero? 348 Coordinate Matrices and Rank, 350 Characterizing Vectors in a Tensor Product, 354 Defining Linear Transformations on a Tensor Product, 355 The Tensor Product of Linear Transformations, 357 Change of Base Field, 359 Multilinear Maps and Iterated Tensor Products, 363 Tensor Spaces, 366 Special Multilinear Maps, 371 Graded Algebras, 372 The Symmetric Tensor Algebra, 374 The Antisymmetric Tensor Algebra: The Exterior Product Space, 380 The Determinant, 387 Exercises, 391

15 Positive Solutions to Linear Systems: Convexity and Separation 395 Convex, Closed and Compact Sets, 398 Convex Hulls, 399

Linear and Affine Hyperplanes, 400 Separation, 402 Exercises, 407 16 Affine Geometry, 409 Affine Geometry, 409 Affine Combinations, 41 Affine Hulls, 412 The Lattice of Flats, 413 Affine Independence, 416 Affine Transformations, 417 Projective Geometry, 419 Exercises, 423 17 **Operator Factorizations: QR and Singular Value, 425** The QR Decomposition, 425 Singular Values, 428 The Moore–Penrose Generalized Inverse, 430 Least Squares Approximation, 433 Exercises, 434 18 The Umbral Calculus, 437 Formal Power Series, 437 The Umbral Algebra, 439 Formal Power Series as Linear Operators, 443 Sheffer Sequences, 446 Examples of Sheffer Sequences, 454 Umbral Operators and Umbral Shifts, 456 Continuous Operators on the Umbral Algebra, 458 **Operator Adjoints**, 459 Umbral Operators and Automorphisms of the Umbral Algebra, 460 Umbral Shifts and Derivations of the Umbral Algebra, 465 The Transfer Formulas, 470 A Final Remark, 471 Exercises, 472

References, 473

Index, 475