

---

## Contents

<i>Preface</i> .....	vii
----------------------	-----

---

### Part I Motivation for Preconditioning

---

<b>1 A Finite Element Tutorial</b> .....	3
1.1 Finite element matrices .....	3
1.2 Finite element refinement .....	9
1.3 Coarse-grid approximation .....	10
1.4 The mass (Gram) matrix .....	15
1.5 A “strong” approximation property .....	18
1.6 The coarse-grid correction .....	21
1.7 A f.e. (geometric) two-grid method .....	22
1.8 Element matrices and matrix orderings .....	25
1.9 Element topology .....	29
1.9.1 Main definitions and constructions .....	30
1.9.2 Element faces .....	32
1.9.3 Faces of AEs .....	34
1.9.4 Edges of AEs .....	35
1.9.5 Vertices of AEs .....	36
1.9.6 Nested dissection ordering .....	36
1.9.7 Element agglomeration algorithms .....	37
1.10 Finite element matrices on many processors .....	39
1.11 The mortar method .....	41
1.11.1 Algebraic construction of mortar spaces .....	43
<b>2 A Main Goal</b> .....	49

---

**Part II Block Factorization Preconditioners**


---

<b>3</b>	<b>Two-by-Two Block Matrices and Their Factorization</b> . . . . .	55
3.1	Matrices of two-by-two block form . . . . .	55
3.1.1	Exact block-factorization. Schur complements . . . . .	55
3.1.2	Kato's Lemma . . . . .	60
3.1.3	Convergent iteration in $A$ -norm . . . . .	61
3.2	Approximate block-factorization . . . . .	63
3.2.1	Product iteration matrix formula . . . . .	63
3.2.2	Block-factorizations and product iteration methods . . . . .	65
3.2.3	Definitions of two-level $B_{TL}$ and two-grid $B_{TG}$ preconditioners . . . . .	67
3.2.4	A main identity . . . . .	68
3.2.5	A simple lower-bound estimate . . . . .	70
3.2.6	Sharp upper bound . . . . .	70
3.2.7	The sharp spectral equivalence result . . . . .	72
3.2.8	Analysis of $B_{TL}$ . . . . .	74
3.2.9	Analysis of $B_{TG}$ . . . . .	75
3.3	Algebraic two-grid methods and preconditioners; sufficient conditions for spectral equivalence . . . . .	78
3.4	Classical two-level block-factorization preconditioners . . . . .	81
3.4.1	A general procedure of generating stable block-matrix partitioning . . . . .	84
<b>4</b>	<b>Classical Examples of Block-Factorizations</b> . . . . .	89
4.1	Block-ILU factorizations . . . . .	89
4.2	The $M$ -matrix case . . . . .	92
4.3	Decay rates of inverses of band matrices . . . . .	98
4.4	Algorithms for approximate band inverses . . . . .	103
4.5	Wittum's frequency filtering decomposition . . . . .	109
4.6	Block-ILU factorizations with block-size reduction . . . . .	113
4.7	An alternative approximate block-LU factorization . . . . .	117
4.8	Odd-even modified block-ILU methods . . . . .	122
4.9	A nested dissection (approximate) inverse . . . . .	125
<b>5</b>	<b>Multigrid (MG)</b> . . . . .	129
5.1	From two-grid to multigrid . . . . .	129
5.2	MG as block Gauss-Seidel . . . . .	133
5.3	A MG analysis in general terms . . . . .	134
5.4	The XZ identity . . . . .	140
5.5	Some classical upper bounds . . . . .	144
5.5.1	Variable $V$ -cycle . . . . .	152
5.6	MG with more recursive cycles; $W$ -cycle . . . . .	157
5.6.1	Definition of a $\nu$ -fold MG-cycle; complexity . . . . .	157
5.6.2	AMLI-cycle multigrid . . . . .	158

5.6.3	Analysis of AMLI	159
5.6.4	Complexity of the AMLI-cycle	162
5.6.5	Optimal $W$ -cycle methods	163
5.7	MG and additive MG	165
5.7.1	The BPX-preconditioner	165
5.7.2	Additive representation of MG	166
5.7.3	Additive MG; convergence properties	167
5.7.4	MG convergence based on results for matrix subblocks	174
5.8	Cascadic multigrid	177
5.8.1	Convergence in a stronger norm	182
5.9	The hierarchical basis (HB) method	185
5.9.1	The additive multilevel HB	185
5.9.2	A stable multilevel hierarchical (direct) decomposition	188
5.9.3	Approximation of $L_2$ -projections	192
5.9.4	Construction of bases in the coordinate spaces	195
5.9.5	The approximate wavelet hierarchical basis (or AWHB)	196
<b>6</b>	<b>Topics on Algebraic Multigrid (AMG)</b>	<b>199</b>
6.1	Motivation for the construction of $P$	199
6.2	On the classical AMG construction of $P$	202
6.3	On the constrained trace minimization construction of $P$	205
6.4	On the coarse-grid selection	207
6.5	On the sparsity pattern of $P$	207
6.6	Coarsening by compatible relaxation	208
6.6.1	Smoothing property and compatible relaxation	209
6.6.2	Using inexact projections	211
6.7	The need for adaptive AMG	213
6.8	Smoothing based on “c”–“f” relaxation	214
6.9	AMGe: An element agglomeration AMG	225
6.9.1	Element-based construction of $P$	226
6.9.2	On various norm bounds of $P$	228
6.10	Multivector fitting interpolation	234
6.11	Window-based spectral AMG	235
6.12	Two-grid convergence of vector-preserving AMG	241
6.13	The result of Vaněk, Brezina, and Mandel	249
6.13.1	Null vector-based polynomially smoothed bases	249
6.13.2	Some properties of Chebyshev-like polynomials	252
6.13.3	A general setting for the SA method	255
<b>7</b>	<b>Domain Decomposition (DD) Methods</b>	<b>263</b>
7.1	Nonoverlapping blocks	263
7.2	Boundary extension mappings based on solving special coarse problems	264
7.3	Weakly overlapping blocks	267
7.4	Classical domain-embedding (DE) preconditioners	270

7.5	DE preconditioners without extension mappings . . . . .	272
7.6	Fast solvers for tensor product matrices . . . . .	274
7.7	Schwarz methods . . . . .	280
7.8	Additive Schwarz preconditioners . . . . .	286
7.9	The domain decomposition paradigm of Bank and Holst . . . . .	291
7.9.1	Local error estimates . . . . .	300
7.10	The FAC method and related preconditioning . . . . .	304
7.11	Auxiliary space preconditioning methods . . . . .	313
<b>8</b>	<b>Preconditioning Nonsymmetric and Indefinite Matrices</b> . . . . .	<b>319</b>
8.1	An abstract setting . . . . .	319
8.2	A perturbation point of view . . . . .	323
8.3	Implementation . . . . .	325
<b>9</b>	<b>Preconditioning Saddle-Point Matrices</b> . . . . .	<b>327</b>
9.1	Basic properties of saddle-point matrices . . . . .	327
9.2	S.p.d. preconditioners . . . . .	330
9.2.1	Preconditioning based on “inf–sup” condition . . . . .	332
9.3	Transforming $A$ to a positive definite matrix . . . . .	339
9.4	(Inexact) Uzawa and distributive relaxation methods . . . . .	341
9.4.1	Distributive relaxation . . . . .	341
9.4.2	The Bramble–Pasciak transformation . . . . .	342
9.4.3	A note on two-grid analysis . . . . .	344
9.4.4	Inexact Uzawa methods . . . . .	347
9.5	A constrained minimization approach . . . . .	353
<b>10</b>	<b>Variable-Step Iterative Methods</b> . . . . .	<b>363</b>
10.1	Variable-step (nonlinear) preconditioners . . . . .	363
10.2	Variable-step preconditioned CG method . . . . .	365
10.3	Variable-step multilevel preconditioners . . . . .	371
10.4	Variable-step AMLI-cycle MG . . . . .	372
<b>11</b>	<b>Preconditioning Nonlinear Problems</b> . . . . .	<b>377</b>
11.1	Problem formulation . . . . .	377
11.2	Choosing an accurate initial approximation . . . . .	379
11.3	The inexact Newton algorithm . . . . .	380
<b>12</b>	<b>Quadratic Constrained Minimization Problems</b> . . . . .	<b>385</b>
12.1	Problem formulation . . . . .	385
12.1.1	Projection methods . . . . .	386
12.1.2	A modified projection method . . . . .	389
12.2	Computable projections . . . . .	390
12.3	Dual problem approach . . . . .	391
12.3.1	Dual problem formulation . . . . .	391
12.3.2	Reduced problem formulation . . . . .	393

12.4	A monotone two-grid scheme . . . . .	397
12.4.1	Projected Gauss–Seidel . . . . .	398
12.4.2	Coarse-grid solution . . . . .	398
12.5	A monotone FAS constrained minimization algorithm . . . . .	401

---

**Part III Appendices**

---

<b>A</b>	<b>Generalized Conjugate Gradient Methods</b> . . . . .	407
A.1	A general variational setting for solving nonsymmetric problems . . .	407
A.2	A quick CG guide . . . . .	410
A.2.1	The CG algorithm . . . . .	410
A.2.2	Preconditioning . . . . .	411
A.2.3	Best polynomial approximation property of CG . . . . .	412
A.2.4	A decay rate estimate for $A^{-1}$ . . . . .	412
<b>B</b>	<b>Properties of Finite Element Matrices. Further Details</b> . . . . .	415
B.1	Piecewise linear finite elements . . . . .	415
B.2	A semilinear second-order elliptic PDE . . . . .	429
B.3	Stable two-level HB decomposition of finite element spaces . . . . .	433
B.3.1	A two-level hierarchical basis and related strengthened Cauchy–Schwarz inequality . . . . .	433
B.3.2	On the MG convergence uniform w.r.t. the mesh and jumps in the PDE coefficients . . . . .	438
B.4	Mixed methods for second-order elliptic PDEs . . . . .	439
B.5	Nonconforming elements and Stokes problem . . . . .	448
B.6	F.e. discretization of Maxwell’s equations . . . . .	453
<b>C</b>	<b>Computable Scales of Sobolev Norms</b> . . . . .	457
C.1	$H^s$ -stable decompositions . . . . .	457
C.2	Preliminary facts . . . . .	457
C.3	The main norm equivalence result . . . . .	459
C.4	The uniform coercivity property . . . . .	463
<b>D</b>	<b>Multilevel Algorithms for Boundary Extension Mappings</b> . . . . .	467
<b>E</b>	<b><math>H_0^1</math>-norm Characterization</b> . . . . .	471
E.1	Optimality of the $L_2$ -projections . . . . .	471
E.1.1	$H_0^1$ -stable decompositions of finite element functions . . . . .	475
<b>F</b>	<b>MG Convergence Results for Finite Element Problems</b> . . . . .	477
F.1	Requirements on the multilevel f.e. decompositions for the MG convergence analysis . . . . .	479
F.2	A MG for weighted $H(\text{curl})$ space . . . . .	487
F.2.1	A multilevel decomposition of weighted Nédélec spaces . . .	489
F.3	A multilevel decomposition of div-free Raviart–Thomas spaces . . .	495
F.4	A multilevel decomposition of weighted $H(\text{div})$ -space . . . . .	499

<b>G</b>	<b>Some Auxiliary Inequalities</b> .....	507
	G.1 Kantorovich's inequality .....	507
	G.2 An inequality between powers of matrices .....	508
	G.3 Energy bound of the nodal interpolation operator .....	510
	<i>References</i> .....	513
	<i>Index</i> .....	527