
Contents

Preface	vii
Standard Notation	xiii
0 A Colloquial Survey of Jordan Theory	1
0.1 Origin of the Species	2
0.2 The Jordan River	6
0.3 Links with Lie Algebras and Groups	10
0.4 Links with Differential Geometry	14
0.5 Links with the Real World	16
0.6 Links with the Complex World	22
0.7 Links with the Infinitely Complex World	25
0.8 Links with Projective Geometry	28

Part I A Historical Survey of Jordan Structure Theory

1 Jordan Algebras in Physical Antiquity	39
1.1 The Matrix Interpretation of Quantum Mechanics	39
1.2 The Jordan Program	40
1.3 The Jordan Operations	40
1.4 Digression on Linearization	41
1.5 Back to the Bullet	42
1.6 The Jordan Axioms	43
1.7 The First Example: Full Algebras	45
1.8 The Second Example: Hermitian Algebras	46
1.9 The Third Example: Spin Factors	47
1.10 Special and Exceptional	48
1.11 Classification	48

2	Jordan Algebras in the Algebraic Renaissance	51
2.1	Linear Algebras over General Scalars	52
2.2	Categorical Nonsense	53
2.3	Commutators and Associators	55
2.4	Lie and Jordan Algebras	57
2.5	The Three Basic Examples Revisited	58
2.6	Jordan Matrix Algebras with Associative Coordinates	59
2.7	Jordan Matrix Algebras with Alternative Coordinates	60
2.8	The n -Squares Problem	61
2.9	Forms Permitting Composition	62
2.10	Composition Algebras	63
2.11	The Cayley–Dickson Construction and Process	64
2.12	Split Composition Algebras	65
2.13	Classification	67
3	Jordan Algebras in the Enlightenment	69
3.1	Forms of Algebras	69
3.2	Inverses and Isotopes	70
3.3	Nuclear Isotopes	71
3.4	Twisted Involutions	72
3.5	Twisted Hermitian Matrices	73
3.6	Spin Factors	74
3.7	Quadratic Factors	74
3.8	Cubic Factors	76
3.9	Reduced Cubic Factors	78
3.10	Classification	79
4	The Classical Theory	81
4.1	U -Operators	81
4.2	The Quadratic Program	82
4.3	The Quadratic Axioms	83
4.4	Justification	84
4.5	Inverses	85
4.6	Isotopes	86
4.7	Inner Ideals	86
4.8	Nondegeneracy	88
4.9	Radical Remarks	89
4.10	i -Special and i -Exceptional	90
4.11	Artin–Wedderburn–Jacobson Structure Theorem	92
5	The Final Classical Formulation	95
5.1	Algebras with Capacity	95
5.2	Classification	97

6	The Classical Methods	99
6.1	Peirce Decompositions	99
6.2	Coordinatization	100
6.3	The Coordinates	102
6.4	Minimal Inner Ideals	102
6.5	Capacity	104
6.6	Capacity Classification	104
7	The Russian Revolution	107
7.1	The Lull Before the Storm	107
7.2	The First Tremors	108
7.3	The Main Quake	109
7.4	Aftershocks	111
8	Zel'manov's Exceptional Methods	114
8.1	I-Finiteness	114
8.2	Absorbers	116
8.3	Modular Inner Ideals	117
8.4	Primitivity	118
8.5	The Heart	119
8.6	Spectra	120
8.7	Comparing Spectra	122
8.8	Big Resolvents	123
8.9	Semiprimitive Imbedding	124
8.10	Ultraproducts	125
8.11	Prime Dichotomy	127

Part II The Classical Theory

1	The Category of Jordan Algebras	132
1.1	Categories	132
1.2	The Category of Linear Algebras	133
1.3	The Category of Unital Algebras	136
1.4	Unitalization	137
1.5	The Category of Algebras with Involution	139
1.6	Nucleus, Center, and Centroid	141
1.7	Strict Simplicity	144
1.8	The Category of Jordan Algebras	146
1.9	Problems for Chapter 1	150
2	The Category of Alternative Algebras	153
2.1	The Category of Alternative Algebras	153
2.2	Nuclear Involutions	154
2.3	Composition Algebras	155

2.4	Split Composition Algebras	157
2.5	The Cayley–Dickson Construction	160
2.6	The Hurwitz Theorem	164
2.7	Problems for Chapter 2	167
3	Three Special Examples	168
3.1	Full Type	168
3.2	Hermitian Type	171
3.3	Quadratic Form Type	176
3.4	Reduced Spin Factors	180
3.5	Problems for Chapter 3	183
4	Jordan Algebras of Cubic Forms	186
4.1	Cubic Maps	187
4.2	The General Construction	188
4.3	The Jordan Cubic Construction	191
4.4	The Freudenthal Construction	193
4.5	The Tits Constructions	195
4.6	Problems for Chapter 4	197
5	Two Basic Principles	199
5.1	The Macdonald and Shirshov–Cohn Principles	199
5.2	Fundamental Formulas	200
5.3	Nondegeneracy	205
5.4	Problems for Chapter 5	209
6	Inverses	211
6.1	Jordan Inverses	211
6.2	von Neumann and Nuclear Inverses	217
6.3	Problems for Chapter 6	219
7	Isotopes	220
7.1	Nuclear Isotopes	220
7.2	Jordan Isotopes	222
7.3	Quadratic Factor Isotopes	224
7.4	Cubic Factor Isotopes	226
7.5	Matrix Isotopes	228
7.6	Problems for Chapter 7	232
8	Peirce Decomposition	235
8.1	Peirce Decompositions	235
8.2	Peirce Multiplication Rules	239
8.3	Basic Examples of Peirce Decompositions	240
8.4	Peirce Identity Principle	245
8.5	Problems for Chapter 8	246

9	Off-Diagonal Rules	247
9.1	Peirce Specializations	247
9.2	Peirce Quadratic Forms	250
9.3	Problems for Chapter 9	252
10	Peirce Consequences	253
10.1	Diagonal Consequences	253
10.2	Diagonal Isotopes	255
10.3	Problems for Chapter 10	258
11	Spin Coordinatization	259
11.1	Spin Frames	259
11.2	Diagonal Spin Consequences	262
11.3	Strong Spin Coordinatization	263
11.4	Spin Coordinatization	265
11.5	Problems for Chapter 11	267
12	Hermitian Coordinatization	268
12.1	Cyclic Frames	268
12.2	Diagonal Hermitian Consequences	270
12.3	Strong Hermitian Coordinatization	272
12.4	Hermitian Coordinatization	274
13	Multiple Peirce Decompositions	278
13.1	Decomposition	278
13.2	Recovery	282
13.3	Multiplication	282
13.4	The Matrix Archetype	284
13.5	The Peirce Principle	286
13.6	Modular Digression	288
13.7	Problems for Chapter 13	290
14	Multiple Peirce Consequences	292
14.1	Jordan Coordinate Conditions	292
14.2	Peirce Specializations	294
14.3	Peirce Quadratic Forms	295
14.4	Connected Idempotents	296
15	Hermitian Symmetries	301
15.1	Hermitian Frames	301
15.2	Hermitian Symmetries	303
15.3	Problems for Chapter 15	307
16	The Coordinate Algebra	308
16.1	The Coordinate Triple	308

17	Jacobson Coordinatization	312
	17.1 Strong Coordinatization	312
	17.2 General Coordinatization	315
18	Von Neumann Regularity	318
	18.1 vNr Pairing	318
	18.2 Structural Pairing	321
	18.3 Problems for Chapter 18	324
19	Inner Simplicity	325
	19.1 Simple Inner Ideals	325
	19.2 Minimal Inner Ideals	327
	19.3 Problems for Chapter 19	329
20	Capacity	330
	20.1 Capacity Existence	330
	20.2 Connected Capacity	331
	20.3 Problems for Chapter 20	334
21	Herstein–Kleinfeld–Osborn Theorem	335
	21.1 Alternative Algebras Revisited	335
	21.2 A Brief Tour of the Alternative Nucleus	338
	21.3 Herstein–Kleinfeld–Osborn Theorem	341
	21.4 Problems for Chapter 21	345
22	Osborn’s Capacity 2 Theorem	348
	22.1 Commutators	348
	22.2 Capacity Two	351
23	Classical Classification	356
	23.1 Capacity $n \geq 3$	356

Part III Zel’manov’s Exceptional Theorem

1	The Radical	362
	1.1 Invertibility	363
	1.2 Structurality	364
	1.3 Quasi-Invertibility	366
	1.4 Proper Quasi-Invertibility	369
	1.5 Elemental Characterization	374
	1.6 Radical Inheritance	375
	1.7 Radical Surgery	376
	1.8 Problems for Chapter 1	379

2	Begetting and Bounding Idempotents	381
2.1	I-gene	381
2.2	Algebraic Implies I-Genic	383
2.3	I-genic Nilness	384
2.4	I-Finiteness	384
2.5	Problems for Chapter 2	387
3	Bounded Spectra Beget Capacity	388
3.1	Spectra	388
3.2	Bigness	390
3.3	Evaporating Division Algebras	392
3.4	Spectral Bounds and Capacity	392
3.5	Problems for Chapter 3	395
4	Absorbers of Inner Ideals	397
4.1	Linear Absorbers	397
4.2	Quadratic Absorbers	400
4.3	Absorber Nilness	403
4.4	Problems for Chapter 4	409
5	Primitivity	410
5.1	Modularity	410
5.2	Primitivity	413
5.3	Semiprimitivity	415
5.4	Imbedding Nondegenerates in Semiprimitives	416
5.5	Problems for Chapter 5	420
6	The Primitive Heart	422
6.1	Hearts and Spectra	422
6.2	Primitive Hearts	424
6.3	Problems for Chapter 6	425
7	Filters and Ultrafilters	427
7.1	Filters in General	427
7.2	Filters from Primes	428
7.3	Ultimate Filters	430
7.4	Problems for Chapter 7	432
8	Ultraproducts	433
8.1	Ultraproducts	433
8.2	Examples	435
8.3	Problems for Chapter 8	439

9	The Final Argument	440
9.1	Dichotomy	440
9.2	The Prime Dichotomy	441
9.3	Problems for Chapter 9	443

Part IV Appendices

A	Cohn's Special Theorems	447
A.1	Free Gadgets	447
A.2	Cohn Symmetry	449
A.3	Cohn Speciality	450
A.4	Problems for Appendix A	454
B	Macdonald's Theorem	455
B.1	The Free Jordan Algebra	455
B.2	Identities	458
B.3	Normal Form for Multiplications	461
B.4	The Macdonald Principles	466
B.5	Albert i -Exceptionality	469
B.5.1	Nonvanishing of G_9	471
B.5.2	Nonvanishing of G_8	472
B.5.3	Nonvanishing of T_{11}	473
B.6	Problems for Appendix B	475
C	Jordan Algebras of Degree 3	476
C.1	Jordan Matrix Algebras	476
C.2	The General Construction	480
C.3	The Freudenthal Construction	487
C.4	The Tits Constructions	490
C.5	Albert Division Algebras	498
C.6	Problems for Appendix C	500
D	The Jacobson–Bourbaki Density Theorem	501
D.1	Semisimple Modules	501
D.2	The Jacobson–Bourbaki Density Theorem	504
E	Hints	506
E.1	Hints for Part II	506
E.2	Hints for Part III	517
E.3	Hints for Part IV	521

Part V Indexes

A	Index of Collateral Readings	524
	A.1 Foundational Readings	524
	A.2 Readings in Applications	527
	A.3 Historical Perusals	529
B	Pronouncing Index of Names	531
C	Index of Notations	536
D	Index of Statements	545
E	Index of Definitions	555