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General Notation

While it is generally the case that the notation special to each chapter is independent of that in the remainder of the book, there is much that is common to the entire book. These items are listed below.

Euclidean Spaces, Integers, etc.

The collection of all real (nonnegative real) numbers is denoted by $\mathbb{R}$ ($\mathbb{R}_+$), positive (nonnegative) integers by $\mathbb{N}$ ($\mathbb{N}_0$ and/or $\mathbb{Z}_+$). Finally, the rationals (nonnegative rationals) are written as $\mathbb{Q}$ ($\mathbb{Q}_+$). The latter spaces are endowed with their standard Borel topologies and Borel σ-fields which will also be referred to as Borel fields. Recall that the Borel field on a topological space $\mathbb{X}$ is the σ-field generated by all open subsets of $\mathbb{X}$; that is, the smallest σ-field that contains all open subsets of $\mathbb{X}$.

The sequence space ℓ^p is the usual one: For any $p > 0$, ℓ^p designates the collection of all sequences $a = (a_k;\ k \geq 1)$ such that $\sum_k |a_k|^p < \infty$. As usual, ℓ^∞ stands for all bounded sequences. When $\infty > p \geq 1$, these ℓ^p spaces can be normed by $\|a\|_{\ell^p} = \{\sum_k |a_k|^p\}^{1/p}$. Using these norms in turn, we can norm the Euclidean space $\mathbb{R}^k$ in various ways. Throughout, we will use the following two norms: (a) the ℓ^∞ norm, which is $|x| = \max_{1\leq j\leq k} |x^{(j)}|$, for $x \in \mathbb{R}^k$; and (b) the ℓ^2 norm, which is $\|x\| = \{\sum_{j=1}^k |x^{(j)}|^2\}^{\frac{1}{2}}$.

Product Spaces

Throughout, the "dimension" numbers d and N are reserved for the spatial and temporal dimension, respectively.

Given any two sets F and T, the set F^T is defined as the collection of all functions $f : T \mapsto F$. When F is a topological space, F^T is often endowed with the product topology; cf. Appendix D. If $m \in \mathbb{N}$, F^m is the usual product space

$$F^m = \overbrace{F \times \cdots \times F}^{m \text{ times}}.$$

This, too, is often endowed with the product topology if and when F is topological.

Throughout, the ith coordinate of any point $s \in \mathbb{R}^m$ is written as $s^{(i)}$. We need the following special order structure on the $\mathbb{R}^m$: Whenever $s, t \in \mathbb{R}^m$, we write $s \preccurlyeq t$ ($s \prec t$) when for $i = 1, \ldots, m$, $s^{(i)} \leq t^{(i)}$ ($s^{(i)} < t^{(i)}$). Occasionally, we may write this as $t \succcurlyeq s$ (or $t \succ s$). Whenever $s, t \in \mathbb{R}^m$, $s \curlywedge t$ designates the point whose ith coordinate is $s^{(i)} \wedge t^{(i)}$ for all $i = 1, \ldots, m$.

If $s \preccurlyeq t$, then $[s, t] = \prod_{j=1}^m [s^{(j)}, t^{(j)}]$. We will refer to $[s, t]$ as a **rectangle** (or an m-dimensional rectangle). When this rectangle is of the form $[s, s + (r, \ldots, r)]$ for some $r \in \mathbb{R}^1_+$ and $s \in \mathbb{R}^m$, then it is a (hyper)**cube**. If $s \prec t$, one can similarly define $]s, t[$, $[s, t[$, and $]s, t[$. For instance, $]s, t] = \prod_{j=1}^m]s^{(j)}, t^{(j)}]$. All subsets of $\mathbb{R}^m$ automatically inherit the order structure of $\mathbb{R}^m$. This way, $s \preccurlyeq t$ makes the same sense in $\mathbb{N}^k$ as it does in $\mathbb{R}^k$, for instance.

Probability and Measure Theory

Unless it is stated to the contrary, the underlying probability space is nearly always denoted by $(\Omega, \mathcal{G}, \mathbb{P})$, where Ω is the so-called sample space, $\mathcal{G}$ is a σ-field of subsets of Ω, and $\mathbb{P}$ is a probability measure on $\mathcal{G}$. Unless it is specifically stated otherwise, the corresponding expectation operator is always denoted by $\mathbb{E}$.

While intersections of σ-fields are themselves σ-fields, their unions are not always. Thus, when $\mathcal{F}_1$ and $\mathcal{F}_2$ are two σ-fields, we write $\mathcal{F}_1 \vee \mathcal{F}_2$ to mean the smallest σ-field that contains $\mathcal{F}_1 \cup \mathcal{F}_2$. More generally, for any index set $\mathbb{A}$, we write $\vee_{\alpha \in \mathbb{A}} \mathcal{F}_\alpha$ for the smallest σ-field that contains all of the σ-fields $(\mathcal{F}_\alpha;\ \alpha \in \mathbb{A})$.

An important function is the indicator function (called the characteristic function in the analysis literature): In *any* space for *any* set A in that space, $\mathbb{1}_A$ denotes the function $x \mapsto \mathbb{1}_A(x)$ that is defined by $\mathbb{1}_A(x) = 1$ if $x \in A$ and $\mathbb{1}_A(x) = 0$ if $x \notin A$. In particular, if A is an event in the probability space, $\mathbb{1}_A$ is the indicator function of the event A.

Throughout, "a.s." is treated synonymously to "almost surely," "$\mathbb{P}$-almost everywhere," "almost sure," or "$\mathbb{P}$-almost sure," depending on which is more applicable.

Both "iid" and "i.i.d." stand for "independent, identically distributed."

When μ is a (nonnegative) measure on a measure space $(\Omega, \mathcal{F})$, $L^p(\mu)$ denotes the collection of all real-valued, p-times μ-integrable functions on Ω. In particular, we use this notation quite often when μ is a probability measure $\mathbb{P}$. In this case, we can interpret $L^p(\mathbb{P})$ as the collection of all random variables whose absolute value has p finite moments. (You should recall that L^p spaces are in fact spaces of equivalence relations, where we say that f and g are equivalent when they are equivalent μ-almost everywhere.)

A special case is made for the L^p spaces with respect to Lebesgue's measure (on Euclidean spaces): When $E \subset \mathbb{R}^k$ is Borel (or more generally, Lebesgue) measurable, $L^p(E)$ (or sometimes L^pE) denotes the collection of all p-times continuously differentiable functions that map E into $\mathbb{R}$. For instance, we write $L^p[0,1]$ and/or $L^p([0,1])$ for the collection of (equivalence classes of) all p-times integrable functions on $[0,1]$.

Depending on the point that is being made, a stochastic process $(X_t : t \in T)$ (where T is some indexing set) is identified with the "randomly chosen function" $t \mapsto X_t$.

Throughout, Leb denotes Lebesgue's measure regardless of the dimension of the underlying Euclidean space.