

---

# Contents

<b>1</b>	<b>Introduction</b> . . . . .	1
1.1	Why piecewise smooth? . . . . .	1
1.2	Impact oscillators . . . . .	3
1.2.1	Case study I: A one-degree-of-freedom impact oscillator . . . . .	6
1.2.2	Periodic motion . . . . .	13
1.2.3	What do we actually see? . . . . .	18
1.2.4	Case study II: A bilinear oscillator . . . . .	26
1.3	Other examples of piecewise-smooth systems . . . . .	28
1.3.1	Case study III: Relay control systems . . . . .	28
1.3.2	Case study IV: A dry-friction oscillator . . . . .	32
1.3.3	Case study V: A DC–DC converter . . . . .	34
1.4	Non-smooth one-dimensional maps . . . . .	39
1.4.1	Case study VI: A simple model of irregular heartbeats . . . . .	39
1.4.2	Case study VII: A square-root map . . . . .	42
1.4.3	Case study VIII: A continuous piecewise-linear map . . . . .	44
<b>2</b>	<b>Qualitative theory of non-smooth dynamical systems</b> . . . . .	47
2.1	Smooth dynamical systems . . . . .	47
2.1.1	Ordinary differential equations (flows) . . . . .	49
2.1.2	Iterated maps . . . . .	53
2.1.3	Asymptotic stability . . . . .	58
2.1.4	Structural stability . . . . .	59
2.1.5	Periodic orbits and Poincaré maps . . . . .	63
2.1.6	Bifurcations of smooth systems . . . . .	67
2.2	Piecewise-smooth dynamical systems . . . . .	71
2.2.1	Piecewise-smooth maps . . . . .	71
2.2.2	Piecewise-smooth ODEs . . . . .	73
2.2.3	Filippov systems . . . . .	75
2.2.4	Hybrid dynamical systems . . . . .	78
2.3	Other formalisms for non-smooth systems . . . . .	83
2.3.1	Complementarity systems . . . . .	83

2.3.2	Differential inclusions	88
2.3.3	Control systems	91
2.4	Stability and bifurcation of non-smooth systems	93
2.4.1	Asymptotic stability	94
2.4.2	Structural stability and bifurcation	96
2.4.3	Types of discontinuity-induced bifurcations	100
2.5	Discontinuity mappings	103
2.5.1	Transversal intersections; a motivating calculation	105
2.5.2	Transversal intersections; the general case	107
2.5.3	Non-transversal (grazing) intersections	111
2.6	Numerical methods	114
2.6.1	Direct numerical simulation	115
2.6.2	Path-following	118
<b>3</b>	<b>Border-collision in piecewise-linear continuous maps</b>	<b>121</b>
3.1	Locally piecewise-linear continuous maps	121
3.1.1	Definitions	124
3.1.2	Possible dynamical scenarios	125
3.1.3	Border-collision normal form map	127
3.2	Bifurcation of the simplest orbits	128
3.2.1	A general classification theorem	128
3.2.2	Notation for bifurcation classification	131
3.3	Equivalence of border-collision classification methods	137
3.3.1	Observer canonical form	137
3.3.2	Proof of Theorem 3.1	140
3.4	One-dimensional piecewise-linear maps	143
3.4.1	Periodic orbits of the map	145
3.4.2	Bifurcations between higher modes	147
3.4.3	Robust chaos	149
3.5	Two-dimensional piecewise-linear normal form maps	154
3.5.1	Border-collision scenarios	155
3.5.2	Complex bifurcation sequences	157
3.6	Maps that are noninvertible on one side	159
3.6.1	Robust chaos	159
3.6.2	Numerical examples	164
3.7	Effects of nonlinear perturbations	169
<b>4</b>	<b>Bifurcations in general piecewise-smooth maps</b>	<b>171</b>
4.1	Types of piecewise-smooth maps	171
4.2	Piecewise-smooth discontinuous maps	174
4.2.1	The general case	174
4.2.2	One-dimensional discontinuous maps	176
4.2.3	Periodic behavior: $l = -1$ , $\nu_1 > 0$ , $\nu_2 < 1$	180
4.2.4	Chaotic behavior: $l = -1$ , $\nu_1 > 0$ , $1 < \nu_2 < 2$	185
4.3	Square-root maps	188

4.3.1	The one-dimensional square-root map . . . . .	188
4.3.2	Quasi one-dimensional behavior . . . . .	193
4.3.3	Periodic orbits bifurcating from the border-collision . . . . .	199
4.3.4	Two-dimensional square-root maps . . . . .	205
4.4	Higher-order piecewise-smooth maps . . . . .	210
4.4.1	Case I: $\gamma = 2$ . . . . .	211
4.4.2	Case II: $\gamma = 3/2$ . . . . .	213
4.4.3	Period-adding scenarios . . . . .	214
4.4.4	Location of the saddle-node bifurcations . . . . .	217
<b>5</b>	<b>Boundary equilibrium bifurcations in flows</b> . . . . .	<b>219</b>
5.1	Piecewise-smooth continuous flows . . . . .	219
5.1.1	Classification of simplest BEB scenarios . . . . .	221
5.1.2	Existence of other attractors . . . . .	225
5.1.3	Planar piecewise-smooth continuous systems . . . . .	226
5.1.4	Higher-dimensional systems . . . . .	229
5.1.5	Global phenomena for persistent boundary equilibria . . . . .	232
5.2	Filippov flows . . . . .	233
5.2.1	Classification of the possible cases . . . . .	235
5.2.2	Planar Filippov systems . . . . .	237
5.2.3	Some global and non-generic phenomena . . . . .	242
5.3	Equilibria of impacting hybrid systems . . . . .	245
5.3.1	Classification of the simplest BEB scenarios . . . . .	246
5.3.2	The existence of other invariant sets . . . . .	249
<b>6</b>	<b>Limit cycle bifurcations in impacting systems</b> . . . . .	<b>253</b>
6.1	The impacting class of hybrid systems . . . . .	253
6.1.1	Examples . . . . .	255
6.1.2	Poincaré maps related to hybrid systems . . . . .	261
6.2	Discontinuity mappings near grazing . . . . .	265
6.2.1	The geometry near a grazing point . . . . .	266
6.2.2	Approximate calculation of the discontinuity mappings . . . . .	271
6.2.3	Calculating the PDM . . . . .	271
6.2.4	Approximate calculation of the ZDM . . . . .	273
6.2.5	Derivation of the ZDM and PDM using Lie derivatives . . . . .	274
6.3	Grazing bifurcations of periodic orbits . . . . .	279
6.3.1	Constructing compound Poincaré maps . . . . .	280
6.3.2	Unfolding the dynamics of the map . . . . .	284
6.3.3	Examples . . . . .	285
6.4	Chattering and the geometry of the grazing manifold . . . . .	295
6.4.1	Geometry of the stroboscopic map . . . . .	295
6.4.2	Global behavior of the grazing manifold $\mathcal{G}$ . . . . .	296
6.4.3	Chattering and the set $G^{(\infty)}$ . . . . .	299
6.5	Multiple collision bifurcation . . . . .	302

<b>7</b>	<b>Limit cycle bifurcations in piecewise-smooth flows</b> . . . . .	307
7.1	Definitions and examples . . . . .	307
7.2	Grazing with a smooth boundary . . . . .	318
7.2.1	Geometry near a grazing point . . . . .	319
7.2.2	Discontinuity mappings at grazing . . . . .	321
7.2.3	Grazing bifurcations of periodic orbits . . . . .	325
7.2.4	Examples . . . . .	327
7.2.5	Detailed derivation of the discontinuity mappings . . . . .	334
7.3	Boundary-intersection crossing bifurcations . . . . .	340
7.3.1	The discontinuity mapping in the general case . . . . .	341
7.3.2	Derivation of the discontinuity mapping in the corner-collision case . . . . .	346
7.3.3	Examples . . . . .	347
<b>8</b>	<b>Sliding bifurcations in Filippov systems</b> . . . . .	355
8.1	Four possible cases . . . . .	355
8.1.1	The geometry of sliding bifurcations . . . . .	356
8.1.2	Normal form maps for sliding bifurcations . . . . .	359
8.2	Motivating example: a relay feedback system . . . . .	364
8.2.1	An adding-sliding route to chaos . . . . .	366
8.2.2	An adding-sliding bifurcation cascade . . . . .	368
8.2.3	A grazing-sliding cascade . . . . .	370
8.3	Derivation of the discontinuity mappings . . . . .	373
8.3.1	Crossing-sliding bifurcation . . . . .	375
8.3.2	Grazing-sliding bifurcation . . . . .	377
8.3.3	Switching-sliding bifurcation . . . . .	381
8.3.4	Adding-sliding bifurcation . . . . .	382
8.4	Mapping for a whole period: normal form maps . . . . .	383
8.4.1	Crossing-sliding bifurcation . . . . .	384
8.4.2	Grazing-sliding bifurcation . . . . .	390
8.4.3	Switching-sliding bifurcation . . . . .	393
8.4.4	Adding-sliding bifurcation . . . . .	395
8.5	Unfolding the grazing-sliding bifurcation . . . . .	396
8.5.1	Non-sliding period-one orbits . . . . .	396
8.5.2	Sliding orbit of period-one . . . . .	397
8.5.3	Conditions for persistence or a non-smooth fold . . . . .	399
8.5.4	A dry-friction example . . . . .	399
8.6	Other cases . . . . .	403
8.6.1	Grazing-sliding with a repelling sliding region — catastrophe . . . . .	403
8.6.2	Higher-order sliding . . . . .	404

<b>9</b>	<b>Further applications and extensions</b> . . . . .	409
9.1	Experimental impact oscillators: noise and parameter sensitivity . . . . .	409
9.1.1	Noise . . . . .	410
9.1.2	An impacting pendulum: experimental grazing bifurcations . . . . .	412
9.1.3	Parameter uncertainty . . . . .	419
9.2	Rattling gear teeth: the similarity of impacting and piecewise-smooth systems . . . . .	422
9.2.1	Equations of motion . . . . .	423
9.2.2	An illustrative case . . . . .	425
9.2.3	Using an impacting contact model . . . . .	426
9.2.4	Using a piecewise-linear contact model . . . . .	431
9.3	A hydraulic damper: non-smooth invariant tori . . . . .	434
9.3.1	The model . . . . .	436
9.3.2	Grazing bifurcations . . . . .	438
9.3.3	A grazing bifurcation analysis for invariant tori . . . . .	441
9.4	Two-parameter sliding bifurcations in friction oscillators . . . . .	448
9.4.1	A degenerate crossing-sliding bifurcation . . . . .	449
9.4.2	Fold bifurcations of grazing-sliding limit cycles . . . . .	453
9.4.3	Two simultaneous grazings . . . . .	455
	<b>References</b> . . . . .	459
	<b>Index</b> . . . . .	475