Contents

Preface ..... **vii**

1 Geometry **1**

1.1 Lines, angles and trigonometry 4
1.1.1 Points and straight lines 4
1.1.2 Angles 4
1.1.3 Trigonometry 5

1.2 Circles 9
1.2.1 Properties of circles 9
1.2.2 Ellipses 10

1.3 Triangles 11
1.3.1 Types of triangle 11
1.3.2 Similar triangles 11
1.3.3 Congruent triangles 12
1.3.4 Theorem of Pythagoras 12
1.3.5 Internal and external angles 13
1.3.6 Sine, cosine and tangent rules 13
1.3.7 Area of a triangle 13
1.3.8 Inscribed and circumscribed circles 14
1.3.9 Centroid of a triangle 15
1.3.10 Spherical trigonometry 15

1.4 Quadrilaterals 16

1.5 Polygons 19
1.5.1 Internal and external angles of a polygon 19
1.5.2 Alternate internal angles of a cyclic polygon 19
1.5.3 Area of a regular polygon 19

1.6 Three-dimensional objects 21
1.6.1 Prisms 21
1.6.2 Pyramids 21
1.6.3 Cylinders 22
1.6.4 Cones 22
1.6.5 Spheres 22

1.6.6 Tori 23
1.6.7 Platonic solids 23

1.7 Coordinate systems 26
1.7.1 Cartesian coordinates in $\mathbb{R}^2$ 26
1.7.2 Cartesian coordinates in $\mathbb{R}^3$ 26
1.7.3 Polar coordinates 27
1.7.4 Cylindrical coordinates 27
1.7.5 Spherical coordinates 28

1.8 Vectors 29
1.8.1 Vector between two points 29
1.8.2 Scaling a vector 29
1.8.3 Reversing a vector 29
1.8.4 Unit Cartesian vectors 29
1.8.5 Algebraic notation for a vector 29
1.8.6 Magnitude of a vector 30
1.8.7 Normalizing a vector to a unit length 30
1.8.8 Vector addition/subtraction 30
1.8.9 Compound scalar multiplication 30
1.8.10 Position vector 30
1.8.11 Scalar (dot) product 30
1.8.12 Angle between two vectors 31
1.8.13 Vector (cross) product 31
1.8.14 The commutative law does not hold 31
1.8.15 Scalar triple product 31
1.8.16 Vector triple product 32
1.8.17 Vector normal to a triangle 32
1.8.18 Area of a triangle 32

1.9 Quaternions 33
1.9.1 Definition of a quaternion 33
1.9.2 Equal quaternions 33
1.9.3 Quaternion addition and subtraction 33
1.9.4 Quaternion multiplication 33
1.9.5 Magnitude of a quaternion 34
1.9.6 The inverse quaternion 34
1.9.7 Rotating a vector 34
1.9.8 Quaternion as a matrix 34

1.10 Transformations 35
1.10.1 Scaling relative to the origin in $\mathbb{R}^2$ 35
1.10.2 Scaling relative to a point in $\mathbb{R}^2$ 35
1.10.3 Translation in $\mathbb{R}^2$ 35
1.10.4 Rotation about the origin in $\mathbb{R}^2$ 35
1.10.5 Rotation about a point in $\mathbb{R}^2$ 36
1.10.6 Shearing along the x-axis in $\mathbb{R}^2$ 36
1.10.7 Shearing along the y-axis in $\mathbb{R}^2$ 36
1.10.8 Reflection about the x-axis in $\mathbb{R}^2$ 36
1.10.9 Reflection about the y-axis in $\mathbb{R}^2$ 36

1.10.10 Reflection about a line parallel with the x-axis in $\mathbb{R}^2$ 37
1.10.11 Reflection about a line parallel with the y-axis in $\mathbb{R}^2$ 37
1.10.12 Translated change of axes in $\mathbb{R}^2$ 37
1.10.13 Rotated change of axes in $\mathbb{R}^2$ 37
1.10.14 The identity matrix in $\mathbb{R}^2$ 37
1.10.15 Scaling relative to the origin in $\mathbb{R}^3$ 38
1.10.16 Scaling relative to a point in $\mathbb{R}^3$ 38
1.10.17 Translation in $\mathbb{R}^3$ 38
1.10.18 Rotation about the x-axis in $\mathbb{R}^3$ 38
1.10.19 Rotation about the y-axis in $\mathbb{R}^3$ 39
1.10.20 Rotation about the z-axis in $\mathbb{R}^3$ 39
1.10.21 Rotation about an arbitrary axis in $\mathbb{R}^3$ 39
1.10.22 Reflection about the yz-plane in $\mathbb{R}^3$ 39
1.10.23 Reflection about the zx-plane in $\mathbb{R}^3$ 39
1.10.24 Reflection about the xy-plane in $\mathbb{R}^3$ 40
1.10.25 Reflection about a plane parallel with the yz-plane in $\mathbb{R}^3$ 40
1.10.26 Reflection about a plane parallel with the zx-plane in $\mathbb{R}^3$ 40
1.10.27 Reflection about a plane parallel with the xy-plane in $\mathbb{R}^3$ 40
1.10.28 Translated change of axes in $\mathbb{R}^3$ 40
1.10.29 Rotated change of axes in $\mathbb{R}^3$ 41
1.10.30 The identity matrix in $\mathbb{R}^3$ 41

1.11 Two-dimensional straight lines 42
1.11.1 Normal form of the straight line equation 42
1.11.2 General form of the straight line equation 42
1.11.3 Hessian normal form of the straight line equation 42
1.11.4 Parametric form of the straight line equation 42
1.11.5 Cartesian form of the straight line equation 43
1.11.6 Straight-line equation from two points 43
1.11.7 Point of intersection of two straight lines 44
1.11.8 Angle between two straight lines 45
1.11.9 Three points lie on a straight line 45
1.11.10 Parallel and perpendicular straight lines 46
1.11.11 Position and distance of a point on a line perpendicular to the origin 46
1.11.12 Position and distance of the nearest point on a line to a point 47
1.11.13 Position of a point reflected in a line 47
1.11.14 Normal to a line through a point 48
1.11.15 Line equidistant from two points 48
1.11.16 Two-dimensional line segment 49

1.12 Lines and circles 51
1.12.1 Line intersecting a circle 51
1.12.2 Touching and intersecting circles 51

1.13 Second degree curves 53
1.13.1 Circle 53
1.13.2 Ellipse 53

1.13.3 Parabola 54
1.13.4 Hyperbola 54

1.14 Three-dimensional straight lines 55
1.14.1 Straight line equation from two points 55
1.14.2 Intersection of two straight lines 55
1.14.3 The angle between two straight lines 55
1.14.4 Three points lie on a straight line 55
1.14.5 Parallel and perpendicular straight lines 56
1.14.6 Position and distance of a point on a line perpendicular to the origin 56
1.14.7 Position and distance of the nearest point on a line to a point 56
1.14.8 Shortest distance between two skew lines 56
1.14.9 Position of a point reflected in a line 57
1.14.10 Normal to a line through a point 57

1.15 Planes 58
1.15.1 Cartesian form of the plane equation 58
1.15.2 General form of the plane equation 58
1.15.3 Hessian normal form of the plane equation 58
1.15.4 Parametric form of the plane equation 59
1.15.5 Converting from the parametric form to the general form 59
1.15.6 Plane equation from three points 59
1.15.7 Plane through a point and normal to a line 60
1.15.8 Plane through two points and parallel to a line 60
1.15.9 Intersection of two planes 60
1.15.10 Intersection of three planes 61
1.15.11 Angle between two planes 61
1.15.12 Angle between a line and a plane 62
1.15.13 Intersection of a line and a plane 62
1.15.14 Position and distance of the nearest point on a plane to a point 62
1.15.15 Reflection of a point in a plane 63
1.15.16 Plane equidistant from two points 63
1.15.17 Reflected ray on a surface 63

1.16 Lines, planes and spheres 64
1.16.1 Line intersecting a sphere 64
1.16.2 Sphere touching a plane 64
1.16.3 Touching spheres 64

1.17 Three-dimensional triangles 66
1.17.1 Point inside a triangle 66
1.17.2 Unknown coordinate value inside a triangle 66

1.18 Parametric curves and patches 67
1.18.1 Parametric curve in $\mathbb{R}^2$ 67
1.18.2 Parametric curve in $\mathbb{R}^3$ 67
1.18.3 Planar patch 67
1.18.4 Modulated surface 68

1.18.5 Quadratic Bézier curve 68
1.18.6 Cubic Bézier curve 68
1.18.7 Quadratic Bézier patch 68
1.18.8 Cubic Bézier patch 69

1.19 Second degree surfaces in standard form 70

2 Examples 73

2.1 Trigonometry 75

2.2 Circles 78

2.3 Triangles 79
2.3.1 Checking for similar triangles 79
2.3.2 Checking for congruent triangles 79
2.3.3 Solving the angles and sides of a triangle 80
2.3.4 Calculating the area of a triangle 81
2.3.5 The center and radius of the inscribed and circumscribed circles for a triangle 82

2.4 Quadrilaterals 84

2.5 Polygons 86

2.6 Three-dimensional objects 88
2.6.1 Cone, cylinder and sphere 88
2.6.2 Conical frustum, spherical segment and torus 88
2.6.3 Tetrahedron 89

2.7 Coordinate systems 90
2.7.1 Cartesian coordinates in $\mathbb{R}^2$ 90
2.7.2 Cartesian coordinates in $\mathbb{R}^3$ 90
2.7.3 Polar coordinates 90
2.7.4 Cylindrical coordinates 91
2.7.5 Spherical coordinates 92

2.8 Vectors 94
2.8.1 Vector between two points 94
2.8.2 Scaling a vector 94
2.8.3 Reversing a vector 94
2.8.4 Magnitude of a vector 94
2.8.5 Normalizing a vector to a unit length 94
2.8.6 Vector addition/subtraction 94
2.8.7 Position vector 95
2.8.8 Scalar (dot) product 95
2.8.9 Angle between two vectors 95
2.8.10 Vector (cross) product 95
2.8.11 Scalar triple product 96
2.8.12 Vector normal to a triangle 96
2.8.13 Area of a triangle 96

2.9 Quaternions 97
2.9.1 Quaternion addition and subtraction 97
2.9.2 Quaternion multiplication 97
2.9.3 Magnitude of a quaternion 97
2.9.4 The inverse quaternion 97
2.9.5 Rotating a vector 97
2.9.6 Quaternion as a matrix 98

2.10 Transformations 99
2.10.1 Scaling relative to the origin in $\mathbb{R}^2$ 99
2.10.2 Scaling relative to a point in $\mathbb{R}^2$ 99
2.10.3 Translation in $\mathbb{R}^2$ 99
2.10.4 Rotation about the origin in $\mathbb{R}^2$ 100
2.10.5 Rotation about a point in $\mathbb{R}^2$ 100
2.10.6 Shearing along the *x*-axis in $\mathbb{R}^2$ 100
2.10.7 Shearing along the *y*-axis in $\mathbb{R}^2$ 101
2.10.8 Reflection about the *x*-axis in $\mathbb{R}^2$ 101
2.10.9 Reflection about the *y*-axis in $\mathbb{R}^2$ 101
2.10.10 Reflection about a line parallel with the *x*-axis in $\mathbb{R}^2$ 102
2.10.11 Reflection about a line parallel with the *y*-axis in $\mathbb{R}^2$ 102
2.10.12 Translated change of axes in $\mathbb{R}^2$ 102
2.10.13 Rotated change of axes in $\mathbb{R}^2$ 103
2.10.14 The identity matrix in $\mathbb{R}^2$ 103
2.10.15 Scaling relative to the origin in $\mathbb{R}^3$ 103
2.10.16 Scaling relative to a point in $\mathbb{R}^3$ 104
2.10.17 Translation in $\mathbb{R}^3$ 104
2.10.18 Rotation about the *x*-axis in $\mathbb{R}^3$ 104
2.10.19 Rotation about the *y*-axis in $\mathbb{R}^3$ 105
2.10.20 Rotation about the *z*-axis in $\mathbb{R}^3$ 105
2.10.21 Rotation about an arbitrary axis in $\mathbb{R}^3$ 105
2.10.22 Reflection about the *yz*-plane in $\mathbb{R}^3$ 106
2.10.23 Reflection about the *zx*-plane in $\mathbb{R}^3$ 106
2.10.24 Reflection about the *xy*-plane in $\mathbb{R}^3$ 106
2.10.25 Reflection about a plane parallel with the *yz*-plane in $\mathbb{R}^3$ 107
2.10.26 Reflection about a plane parallel with the *zx*-plane in $\mathbb{R}^3$ 107
2.10.27 Reflection about a plane parallel with the *xy*-plane in $\mathbb{R}^3$ 107
2.10.28 Translated axes in $\mathbb{R}^3$ 108
2.10.29 Rotated axes in $\mathbb{R}^3$ 108
2.10.30 The identity matrix in $\mathbb{R}^3$ 108

2.11 Two-dimensional straight lines 109
2.11.1 Convert the normal form of the line equation to its general form and the Hessian normal form 109
2.11.2 Derive the unit normal vector and perpendicular from the origin to the line for the line equation 109
2.11.3 Derive the straight-line equation from two points 110
2.11.4 Point of intersection of two straight lines 111
2.11.5 Calculate the angle between two straight lines 112
2.11.6 Test if three points lie on a straight line 113

2.11.7 Test for parallel and perpendicular lines 114
2.11.8 Find the position and distance of the nearest point on a line to the origin 115
2.11.9 Find the position and distance of the nearest point on a line to a point 116
2.11.10 Find the reflection of a point in a line passing through the origin ... 117
2.11.11 Find the reflection of a point in a line 118
2.11.12 Find the normal to a line through a point 119
2.11.13 Find the line equidistant from two points 120
2.11.14 Creating the parametric line equation for a line segment 121
2.11.15 Intersecting two line segments 121

2.12 Lines and circles 123
2.12.1 Line intersecting a circle 123
2.12.2 Touching and intersecting circles 126

2.13 Second degree curves 128
2.13.1 Circle 128
2.13.2 Ellipse 128
2.13.3 Parabola 128
2.13.4 Hyperbola 129

2.14 Three-dimensional straight lines 130
2.14.1 Derive the straight-line equation from two points 130
2.14.2 Intersection of two straight lines 130
2.14.3 Calculate the angle between two straight lines 131
2.14.4 Test if three points lie on a straight line 131
2.14.5 Test for parallel and perpendicular straight lines 132
2.14.6 Find the position and distance of the nearest point on a line to the origin 132
2.14.7 Find the position and distance of the nearest point on a line to a point 132
2.14.8 Find the reflection of a point in a line 133
2.14.9 Find the normal to a line through a point 133
2.14.10 Find the shortest distance between two skew lines 134

2.15 Planes 135
2.15.1 Cartesian form of the plane equation 135
2.15.2 General form of the plane equation 135
2.15.3 Hessian normal form of the plane equation 135
2.15.4 Parametric form of the plane equation 136
2.15.5 Converting a plane equation from parametric form to general form 136
2.15.6 Plane equation from three points 137
2.15.7 Plane through a point and normal to a line 138
2.15.8 Plane through two points and parallel to a line 138
2.15.9 Intersection of two planes 139
2.15.10 Intersection of three planes 141
2.15.11 Angle between two planes 143
2.15.12 Angle between a line and a plane 143

2.15.13 Intersection of a line and a plane 144
2.15.14 Position and distance of the nearest point on a plane to a point . 144
2.15.15 Reflection of a point in a plane 145
2.15.16 Plane equidistant from two points 145
2.15.17 Reflected ray on a surface 146

2.16 Lines, planes and spheres 148
2.16.1 Line intersecting a sphere 148
2.16.2 Sphere touching a plane 149
2.16.3 Touching spheres 150

2.17 Three-dimensional triangles 151
2.17.1 Coordinates of a point inside a triangle 151
2.17.2 Unknown coordinate value inside a triangle 152

2.18 Parametric curves and patches 154
2.18.1 Parametric curves in $\mathbb{R}^2$ 154
2.18.2 Parametric curves in $\mathbb{R}^3$ 158
2.18.3 Planar patch 162
2.18.4 Parametric surfaces in $\mathbb{R}^3$ 163
2.18.5 Quadratic Bézier curve 165
2.18.6 Cubic Bézier curve 165
2.18.7 Quadratic Bézier patch 166
2.18.8 Cubic Bézier patch 167

2.19 Second degree surfaces in standard form 168

3 Proofs 169

3.1 Trigonometry 171
3.1.1 Trigonometric functions and identities 171
3.1.2 Cofunction identities 171
3.1.3 Pythagorean identities 171
3.1.4 Useful trigonometric values 172
3.1.5 Compound angle identities 173
3.1.6 Double-angle identities 175
3.1.7 Multiple-angle identities 175
3.1.8 Functions of the half-angle 176
3.1.9 Functions of the half-angle using the perimeter of a triangle 177
3.1.10 Functions converting to the half-angle tangent form 178
3.1.11 Relationships between sums of functions 180
3.1.12 Inverse trigonometric functions 182

3.2 Circles 183
3.2.1 Angles subtended by the same arc 183
3.2.2 Alternate segment theorem 183
3.2.3 Area of a circle, sector and segment 184
3.2.4 Chord theorem 186
3.2.5 Secant theorem 186
3.2.6 Secant–tangent theorem 186
3.2.7 Area of an ellipse 187

3.3 Triangles 189
3.3.1 Theorem of Pythagoras 189
3.3.2 Properties of triangles 189
3.3.3 Altitude theorem 192
3.3.4 Area of a triangle 193
3.3.5 Internal and external angles of a triangle 196
3.3.6 The medians of a triangle are concurrent at its centroid 196
3.3.7 Radius and center of the inscribed circle for a triangle 198
3.3.8 Radius and center of the circumscribed circle for a triangle 201

3.4 Quadrilaterals 207
3.4.1 Properties of quadrilaterals 207
3.4.2 The opposite sides and angles of a parallelogram are equal 210
3.4.3 The diagonals of a parallelogram bisect each other 210
3.4.4 The diagonals of a square are equal, intersect at right angles and bisect the opposite angles 211
3.4.5 Area of a parallelogram 212
3.4.6 Area of a quadrilateral 212
3.4.7 Area of a general quadrilateral using Heron's formula 214
3.4.8 Area of a trapezoid 216
3.4.9 Radius and center of the circumscribed circle for a rectangle 217

3.5 Polygons 218
3.5.1 The internal angles of a polygon 218
3.5.2 The external angles of a polygon 218
3.5.3 Alternate internal angles of a cyclic polygon 219
3.5.4 Area of a regular polygon 220
3.5.5 Area of a polygon 221
3.5.6 Properties of regular polygons 222

3.6 Three-dimensional objects 224
3.6.1 Volume of a prism 224
3.6.2 Surface area of a rectangular pyramid 225
3.6.3 Volume of a rectangular pyramid 226
3.6.4 Volume of a rectangular pyramidal frustum 227
3.6.5 Volume of a triangular pyramid 227
3.6.6 Surface area of a right cone 228
3.6.7 Surface area of a right conical frustum 228
3.6.8 Volume of a cone 229
3.6.9 Volume of a right conical frustum 230
3.6.10 Surface area of a sphere 230
3.6.11 Volume of a sphere 231
3.6.12 Area and volume of a torus 233
3.6.13 Radii of the spheres associated with the Platonic solids 233
3.6.14 Inner and outer radii for the Platonic solids 238
3.6.15 Dihedral angles for the Platonic solids 242
3.6.16 Surface area and volume of the Platonic solids 246

3.7 Coordinate systems 249
3.7.1 Cartesian coordinates 249
3.7.2 Polar coordinates 249
3.7.3 Cylindrical coordinates 250
3.7.4 Spherical coordinates 250

3.8 Vectors 252
3.8.1 Magnitude of a vector 252
3.8.2 Normalizing a vector to a unit length 252
3.8.3 Scalar (dot) product 252
3.8.4 Commutative law of the scalar product 253
3.8.5 Associative law of the scalar product 253
3.8.6 Angle between two vectors 253
3.8.7 Vector (cross) product 254
3.8.8 The non-commutative law of the vector product 254
3.8.9 The associative law of the vector product 255
3.8.10 Scalar triple product 255

3.9 Quaternions 256
3.9.1 Definition of a quaternion 256

3.10 Transformations 260
3.10.1 Scaling in $\mathbb{R}^2$ 260
3.10.2 Translation in $\mathbb{R}^2$ 261
3.10.3 Rotation in $\mathbb{R}^2$ 261
3.10.4 Shearing in $\mathbb{R}^2$ 262
3.10.5 Reflection in $\mathbb{R}^2$ 263
3.10.6 Change of axes in $\mathbb{R}^2$ 264
3.10.7 Identity matrix in $\mathbb{R}^2$ 265
3.10.8 Scaling in $\mathbb{R}^3$ 265
3.10.9 Translation in $\mathbb{R}^3$ 266
3.10.10 Rotation in $\mathbb{R}^3$ 266
3.10.11 Reflection in $\mathbb{R}^3$ 268
3.10.12 Change of axes in $\mathbb{R}^3$ 270
3.10.13 Identity matrix in $\mathbb{R}^3$ 271

3.11 Two-dimensional straight lines 272
3.11.1 Cartesian form of the line equation 272
3.11.2 Hessian normal form 273
3.11.3 Equation of a line from two points 273
3.11.4 Point of intersection of two straight lines 275
3.11.5 Angle between two straight lines 276
3.11.6 Three points lie on a straight line 277
3.11.7 Parallel and perpendicular straight lines 278
3.11.8 Shortest distance to a line 279
3.11.9 Position and distance of a point on a line perpendicular to the origin 279
3.11.10 Position and distance of the nearest point on a line to a point 280
3.11.11 Position of a point reflected in a line 281
3.11.12 Normal to a line through a point 283

3.11.13 Line equidistant from two points 284
3.11.14 Equation of two-dimensional line segment 285
3.11.15 Point of intersection of two two-dimensional line segments 286

3.12 Lines and circles 288
3.12.1 Line and a circle 288
3.12.2 Touching and intersecting circles 290

3.13 Second degree curves 293
3.13.1 Circle 293
3.13.2 Ellipse 293
3.13.3 Parabola 295
3.13.4 Hyperbola 296

3.14 Three-dimensional straight lines 297
3.14.1 Straight-line equation from two points 297
3.14.2 Intersection of two straight lines 297
3.14.3 Angle between two straight lines 298
3.14.4 Three points lie on a straight line 298
3.14.5 Parallel and perpendicular straight lines 298
3.14.6 Position and distance of a point on a line perpendicular to the origin 299
3.14.7 Position and distance of the nearest point on a line to a point 299
3.14.8 Position of a point reflected in a line 300
3.14.9 Normal to a line through a point 301
3.14.10 Shortest distance between two skew lines 302

3.15 Planes 303
3.15.1 Equation to a plane 303
3.15.2 Plane equation from three points 306
3.15.3 Plane through a point and normal to a line 308
3.15.4 Plane through two points and parallel to a line 308
3.15.5 Intersection of two planes 308
3.15.6 Intersection of three planes 310
3.15.7 Angle between two planes 311
3.15.8 Angle between a line and a plane 311
3.15.9 Intersection of a line and a plane 311
3.15.10 Position and distance of the nearest point on a plane to a point ... 312
3.15.11 Reflection of a point in a plane 313
3.15.12 Plane equidistant from two points 313
3.15.13 Reflected ray on a surface 314

3.16 Lines, planes and spheres 315
3.16.1 Line intersecting a sphere 315
3.16.2 Sphere touching a plane 316
3.16.3 Touching spheres 316

3.17 Three-dimensional triangles 318
3.17.1 Point inside a triangle 318
3.17.2 Unknown coordinate value inside a triangle 318

3.18 Parametric curves and patches 319
3.18.1 Planar surface patch 319
3.18.2 Bézier curves in $\mathbb{R}^2$ and $\mathbb{R}^3$ 319
3.18.3 Bézier surface patch in $\mathbb{R}^3$ 321

4 Glossary 325

5 Bibliography 333

Index 335