

---

## Contents

<b>1</b>	<b>Introduction</b> .....	1
1.1	Conservation Laws for a Continuum Medium.....	1
1.1.1	Conservation of Mass .....	1
1.1.2	Conservation of Momentum.....	2
1.1.3	Conservation of Energy .....	2
1.1.4	Constitutive Relations.....	3
1.2	Governing Equations in Terms of Primitive Variables.....	5
1.2.1	Vector Form.....	5
1.2.2	Component Form in Cartesian Coordinates.....	5
1.2.3	Component Form in Cylindrical Coordinates.....	6
1.2.4	Summary .....	8
1.3	Species Transport Equations.....	8
1.4	Governing Equations in Translating and Rotating Frames of Reference.....	9
1.5	Boundary and Initial Conditions .....	9
1.5.1	General Boundary Conditions .....	10
1.5.2	Free Boundary Conditions.....	11
1.5.3	Moving Interface Conditions.....	12
1.5.4	Phase Change Conditions .....	12
1.6	Governing Equations for Flows Through Porous Media .....	13
1.7	Governing Equations in Conservation Form .....	15
	Exercises .....	17
	References.....	18
<b>2</b>	<b>Discontinuous Finite Element Procedures</b> .....	21
2.1	The Concept of Discontinuous Finite Elements.....	22
2.1.1	Weakly Imposed Cross-element Continuity .....	23
2.1.2	Numerical Boundary Fluxes for Discontinuity.....	25
2.1.3	Boundary Constraint Minimization .....	26
2.1.4	Treatment of Discontinuity for Non-conservative Systems.....	27
2.1.5	Transient Problems .....	28
2.2	Discontinuous Finite Element Formulation .....	29

2.2.1	Integral Formulation .....	29
2.2.2	Time Integration .....	30
2.3	Solution Procedures .....	31
2.4	Advantages and Disadvantages of Discontinuous Finite Element Formulations .....	32
2.4.1	Advantages .....	32
2.4.2	Disadvantages .....	33
2.5	Examples .....	34
	Exercises .....	42
	References .....	42
<b>3</b>	<b>Shape Functions and Elemental Calculations .....</b>	<b>45</b>
3.1	Shape Functions .....	46
3.1.1	1-D Shape Functions .....	46
3.1.2	2-D Shape Functions .....	50
3.1.2.1	Triangular Elements .....	50
3.1.2.2	Quadrilateral Elements .....	54
3.1.3	3-D Shape Functions .....	58
3.1.3.1	Tetrahedral Elements .....	58
3.1.3.2	Hexahedral Elements .....	60
3.2	Construction of Special Elements .....	65
3.2.1	Non-standard Elements .....	65
3.2.2	Construction of Element Shape Functions by Node Collapsing .....	66
3.2.3	Spectral Elements .....	67
3.3	Hierarchical Shape Functions .....	69
3.3.1	1-D Hierarchical Correction .....	69
3.3.2	Canonical Square and Cubic Elements .....	71
3.3.3	Triangular and Tetrahedral Elements .....	73
3.3.4	Obtaining Hierarchical Elements Through Coordinate Transformations .....	77
3.3.5	Orthogonal Mass Matrix Construction .....	78
3.4	Interpolation Error Analysis .....	80
3.4.1	Hilbert Space and Various Error Measures .....	80
3.4.2	Interpolation Error Analysis for 1-D Elements .....	84
3.4.3	Interpolation Error Analysis for 2-D/3-D Elements .....	87
3.5	Numerical Integration .....	88
3.5.1	1-D Numerical Integration .....	88
3.5.2	2-D and 3-D Numerical Integration .....	90
3.5.3	Integration for Triangular and Tetrahedral Elements .....	91
3.6	Elemental Calculations .....	93
3.6.1	Domain Calculations .....	93
3.6.2	Boundary Calculations .....	98
	Exercises .....	102
	References .....	103

<b>4</b>	<b>Conduction Heat Transfer and Potential Flows</b> .....	105
4.1	1-D Steady State Heat Conduction .....	106
4.2	Steady State Heat Conduction in Multidimensions .....	113
4.3	1-D Transient Heat Conduction .....	122
4.3.1	Alternating Upwinding Scheme .....	124
4.3.2	Central Fluxes.....	125
4.3.3	Unified Representation .....	125
4.3.4	Numerical Implementation .....	126
4.3.5	Runge–Kutta Time Integration.....	128
4.3.6	Computational Procedures.....	129
4.4	Transient Heat Conduction in Multidimensions .....	131
4.5	Potential Flows and Flows in Porous Media.....	137
4.6	Selection of Numerical Fluxes.....	138
4.6.1	Stability for Steady State Problems .....	139
4.6.1.1	Stability and Numerical Fluxes .....	139
4.6.1.2	Discontinuous and Mixed Finite Element Formulations .....	142
4.6.2	Stability for Time Dependent Problems .....	144
4.6.2.1	Numerical Fluxes for Transient Problems .....	144
4.6.2.2	Stability Analysis Using Matrix.....	145
4.6.3	Fourier Analysis .....	147
	Exercises.....	152
	References.....	155
<b>5</b>	<b>Convection-dominated Problems</b> .....	157
5.1	Pure Convection Problems.....	158
5.1.1	1-D Pure Convection .....	158
5.1.1.1	Method of Characteristics .....	158
5.1.1.2	Discontinuous Finite Element Formulation .....	162
5.1.2	Pure Convection in Multidimensions .....	166
5.1.3	Stability Analysis.....	169
5.1.3.1	$L^2$ -Stability – Integral Analysis.....	169
5.1.3.2	$L^2$ -Stability – Discretized Analysis.....	171
5.1.3.3	Fourier Analysis.....	176
5.2	Steady State Convection-diffusion.....	181
5.2.1	1-D Problem .....	181
5.2.2	Origin of Oscillatory Stability .....	185
5.2.3	Steady Convection-Diffusion in Multidimensions .....	187
5.3	Transient Convection-diffusion .....	191
5.3.1	Multidimensional Problem .....	191
5.3.2	Stability Analysis.....	195
5.3.2.1	$L^2$ -Stability – Integral Analysis.....	195
5.3.2.2	$L^2$ -Stability – Discretized Analysis.....	198
5.3.2.3	Fourier Analysis.....	199
5.4	Nonlinear Problems .....	202
5.4.1	1-D Inviscid Burgers' Equation.....	202
5.4.1.1	Basic Considerations.....	202

5.4.1.2	Discontinuous Finite Element Formulation .....	204
5.4.2	Multidimensional Inviscid Burgers' Equation.....	209
5.4.2.1	Discontinuous Finite Element Formulation .....	209
5.4.2.2	Characteristic Decomposition .....	211
5.4.3	Higher Order Approximations and TVD Formulations.....	212
5.4.3.1	Concept of Total Variation Diminishing.....	212
5.4.3.2	Flux Limiters.....	214
5.4.3.3	Slope Limiters.....	218
5.4.3.4	TVD-Runge–Kutta Schemes .....	227
5.5	Viscous Burgers' Equations.....	235
5.5.1	1-D Burgers' Equation.....	235
5.5.2	2-D Viscous Burgers' Equation.....	239
	Exercises.....	241
	References.....	243
<b>6</b>	<b>Incompressible Flows .....</b>	<b>247</b>
6.1	Primitive Variable Approach .....	248
6.2	Fractional Step (Projection) Approach .....	258
6.3	Vorticity and Stream Function Approach .....	260
6.4	Coupled Flow and Heat Transfer .....	265
	Exercises.....	269
	References.....	270
<b>7</b>	<b>Compressible Fluid Flows .....</b>	<b>273</b>
7.1	1-D Compressible Flows .....	274
7.1.1	Governing Equations .....	274
7.1.2	Basic Properties of the Euler Equations .....	275
7.1.3	The Rankine–Hugoniot Conditions .....	278
7.1.4	1-D Riemann Solver – Exact Solution.....	281
7.1.5	1-D Riemann Solver – Approximate Solution.....	283
7.1.6	Discontinuous Finite Element Formulation.....	284
7.1.7	Low Order (Finite Volume) Approximations .....	287
7.1.7.1	The Godunov Scheme.....	288
7.1.7.2	The Roe Scheme .....	289
7.1.8	High Order TVD Approximations.....	290
7.1.9	Numerical Examples.....	291
7.2	Multidimensional Inviscid Compressible Flows.....	295
7.2.1	Governing Equations .....	295
7.2.2	Basic Properties of the Split 3-D Euler Equations.....	296
7.2.3	Discontinuous Finite Element Formulation.....	298
7.3	Multidimensional Compressible Viscous Flows.....	299
7.4	ALE Formulation.....	304
7.4.1	ALE Kinematic Description .....	304
7.4.2	Conservation of Mass .....	308
7.4.3	Conservation of Momentum.....	309
7.4.4	Conservation of Energy .....	310
7.4.5	Summary of ALE Equations .....	311

7.4.6	Constitutive Relations.....	311
7.4.7	ALE Description of Compressible Flows.....	311
7.4.8	Discontinuous Finite Element Formulation.....	312
	Exercises.....	313
	References.....	316
<b>8</b>	<b>External Radiative Heat Transfer.....</b>	<b>319</b>
8.1	Integral Equation for Surface Radiation Exchanges.....	320
8.1.1	Governing Equation.....	320
8.1.2	Kernel Functions.....	323
8.2	Discontinuous Galerkin Finite Element Formulation.....	325
8.3	Shadowing Algorithms.....	327
8.3.1	Shadowing Algorithm for 2-D Geometry.....	329
8.3.2	Shadowing Algorithm for Axisymmetric Configurations.....	334
8.3.3	Shadowing Algorithm for 3-D Geometry.....	339
8.4	Coupling with Other Heat Transfer Calculations.....	345
8.4.1	Direct Coupling.....	347
8.4.2	Iterative Coupling.....	348
8.5	Numerical Examples.....	349
	Exercises.....	359
	References.....	360
<b>9</b>	<b>Radiative Transfer In Participating Media.....</b>	<b>363</b>
9.1	Governing Equation and Boundary Conditions.....	364
9.1.1	Radiative Transfer Equation.....	364
9.1.2	Boundary Conditions.....	366
9.2	Approximation Methods.....	367
9.2.1	The Discrete Ordinate Method.....	368
9.2.2	The Spherical Harmonics Method.....	368
9.3	Discontinuous Finite Element Formulation.....	376
9.4	Numerical Implementation.....	389
9.4.1	2-D Calculations.....	389
9.4.2	3-D Calculations.....	396
9.4.3	Integration of the Source Term.....	399
9.4.3.1	The Emitting Contribution.....	399
9.4.3.2	The Scattering Contribution.....	400
9.5	Radiation In Systems of Axisymmetry.....	403
9.5.1	Governing Equation in Cylindrical Coordinates.....	403
9.5.2	Volume Integration.....	405
9.5.3	Surface Integration Over $\Gamma_p$ .....	406
9.5.4	Integration Over $\Gamma_\varphi$ .....	408
9.5.5	Mapping.....	410
9.5.6	Treatment of the Emitting and Scattering Term.....	411
9.6	Use of RTE for External Radiation Calculations.....	412
9.7	Coupling of the Discontinuous Method with Other Methods.....	415
9.8	Constant Element Approximation.....	416
	Exercises.....	424
	References.....	426

<b>10 Free and Moving Boundary Problems</b>	429
10.1 Free and Moving Boundaries	430
10.2 Basic Relations for a Curved Surface	431
10.2.1 Description of a Surface	431
10.2.2 Differential and Integral Relations for Curved Surfaces	436
10.3 Physical Constraints at a Moving Boundary	441
10.3.1 Kinematic Conditions at a Moving Boundary	441
10.3.2 Stress Condition at a Moving Interface	442
10.3.3 Thermal Conditions at a Moving Interface	444
10.4 Moving Grids vs. Fixed Grids for Numerical Solutions	447
10.5 Moving Grid Methods	448
10.5.1 Moving Boundaries Between Fluids	449
10.5.2 Moving Phase Boundaries	454
10.5.2.1 Solid–Liquid Phase Transition	454
10.5.2.2 Liquid–Vapor Phase Transition	457
10.6 Fixed Grid methods	458
10.6.1 Volume of Fluid Method	459
10.6.1.1 Structured Mesh	460
10.6.1.2 Unstructured Mesh	464
10.6.2 The Marker-and-Cell Method	468
10.6.3 The Level Set Method	469
10.6.4 Fixed Grid Methods for Phase Change Problems	473
10.6.4.1 Flow-based Methods	473
10.6.4.2 Enthalpy-based Methods	474
10.7 Phase Field Modeling of Moving Boundaries	476
10.7.1 Basic Ideas of Phase Field Models	476
10.7.2 Governing Equations for Interfacial Phenomena	479
10.7.3 Discontinuous Finite Element Formulation	482
10.7.4 Phase Field Modeling of Microstructure Evolution	483
10.7.4.1 Governing Equations	483
10.7.4.2 Discontinuous Formulation	485
10.7.4.3 Numerical Examples	486
10.7.5 Flow and Orientation Effects on Microstructure Evolution	489
10.7.5.1 Flow Effects on Microstructure Evolution	489
10.7.5.2 Microstructure Evolution During Polycrystalline Solidification	490
Exercises	494
References	498
<b>11 Micro and Nano Scale Fluid Flow and Heat Transfer</b>	501
11.1 Micro Scale Heat Conduction	502
11.1.1 Two-temperature Equations	502
11.1.2 Phonon Scattering Equation	503
11.1.3 Phonon Radiative Transfer Equation	506
11.2 Discontinuous Finite Element Formulation	507
11.3 Micro and Nano Fluid Flow and Heat Transfer	513

11.4	The Boltzmann Transport Equation and Numerical Solution .....	517
11.4.1	The Boltzmann Integral-Differential Equation .....	518
11.4.2	Numerical Solution of the Boltzmann Transport Equation .....	523
11.5	The Boltzman–BGK Equation and Numerical Solution .....	524
11.5.1	The Boltzmann–BGK Equation .....	524
11.5.2	Discontinuous Finite Element Formulation .....	525
11.6	The Lattice Boltzman Equation and Numerical Solution .....	526
11.6.1	Derivation of the Lattice Boltzmann Equation .....	527
11.6.2	Boundary Conditions .....	533
11.6.2.1	Bounce Back Boundary Conditions (No Slip) .....	533
11.6.2.2	Symmetric Boundary Conditions (Free Slip) .....	533
11.6.2.3	Inflow and Outflow Boundary Conditions (No Gradient) .....	534
11.6.2.4	Force Field Conditions .....	534
11.6.2.5	Moving Wall Conditions .....	534
11.6.3	Discontinuous Finite Element Formulation .....	534
	Exercises .....	538
	References .....	540
<b>12</b>	<b>Fluid Flow and Heat Transfer in Electromagnetic Fields .....</b>	<b>543</b>
12.1	Maxwell Equations and Boundary Conditions .....	544
12.1.1	Maxwell Equations .....	544
12.1.2	Boundary Conditions .....	545
12.1.2.1	Interface Boundary Condition .....	545
12.1.2.2	Perfect Conducting Surface .....	546
12.1.2.3	Impedance Condition .....	547
12.1.2.4	Sommerfield Radiation Condition .....	547
12.1.2.5	Symmetry Boundary Condition .....	548
12.2	Maxwell Stresses and Energy Sources .....	548
12.3	Discontinuous Formulation of the Maxwell Equations .....	549
12.3.1	Solution in Time Domain .....	549
12.3.2	Solution in Frequency Domain .....	550
12.3.3	Solution in Other Forms .....	550
12.4	Electroosmotic Flows .....	550
12.4.1	Governing Equations .....	551
12.4.2	Discontinuous Finite Element Formulation .....	553
12.5	Microwave Heating .....	556
12.6	Electrically Deformed Free Surfaces .....	557
12.7	Compressible flows in Magnetic Fields .....	566
	Exercises .....	568
	References .....	569
<b>Index</b>	.....	<b>571</b>