
Contents

1	Classes of Random Trees	1
1.1	Basic Notions	2
1.1.1	Rooted Versus Unrooted trees	2
1.1.2	Plane Versus Non-Plane trees	3
1.1.3	Labelled Versus Unlabelled Trees	3
1.2	Combinatorial Trees	4
1.2.1	Binary Trees	5
1.2.2	Planted Plane Trees	6
1.2.3	Labelled Trees	7
1.2.4	Labelled Plane Trees	8
1.2.5	Unlabelled Trees	8
1.2.6	Unlabelled Plane Trees	9
1.2.7	Simply Generated Trees – Galton-Watson Trees	9
1.3	Recursive Trees	13
1.3.1	Non-Plane Recursive Trees	13
1.3.2	Plane Oriented Recursive Trees	14
1.3.3	Increasing Trees	15
1.4	Search Trees	17
1.4.1	Binary Search Trees	18
1.4.2	Fringe Balanced m -Ary Search Trees	19
1.4.3	Digital Search Trees	21
1.4.4	Tries and Patricia Tries	22
2	Generating Functions	25
2.1	Counting with Generating Functions	26
2.1.1	Generating Functions and Combinatorial Constructions	27
2.1.2	Pólya’s Theory of Counting	33
2.1.3	Lagrange Inversion Formula	36
2.2	Asymptotics with Generating Functions	37
2.2.1	Asymptotic Transfers	38
2.2.2	Functional Equations	43

2.2.3	Asymptotic Normality and Functional Equations	46
2.2.4	Transfer of Singularities	54
2.2.5	Systems of Functional Equations	62
3	Advanced Tree Counting	69
3.1	Generating Functions and Combinatorial Trees	70
3.1.1	Binary and m -Ary Trees	70
3.1.2	Planted Plane Trees	71
3.1.3	Labelled Trees	73
3.1.4	Simply Generated Trees – Galton-Watson Trees	75
3.1.5	Unrooted Trees	77
3.1.6	Trees Embedded in the Plane	81
3.2	Additive Parameters in Trees	82
3.2.1	Simply Generated Trees – Galton-Watson trees	84
3.2.2	Unrooted Trees	87
3.3	Patterns in Trees	90
3.3.1	Planted, Rooted and Unrooted Trees	91
3.3.2	Generating Functions for Planted Rooted Trees	92
3.3.3	Rooted and Unrooted Trees	99
3.3.4	Asymptotic Behavior	101
4	Planar Graphs	107
4.1	Basic Notions	108
4.2	Counting Planar Graphs	110
4.2.1	Outerplanar Graphs	110
4.2.2	Series-Parallel Graphs	118
4.2.3	Quadrangulations and Planar Maps	123
4.2.4	Planar Graphs	130
4.3	Outerplanar Graphs	137
4.3.1	The Degree Distribution of Outerplanar Graphs	137
4.3.2	Vertices of Given Degree in Dissections	141
4.3.3	Vertices of Given Degree in 2-Connected Outerplanar Graphs	146
4.3.4	Vertices of Given Degree in Connected Outerplanar Graphs	147
4.4	Series-Parallel Graphs	150
4.4.1	The Degree Distribution of Series-Parallel Graphs	150
4.4.2	Vertices of Given Degree in Series-Parallel Networks	156
4.4.3	Vertices of Given Degree in 2-Connected Series-Parallel Graphs	158
4.4.4	Vertices of Given Degree in Connected Series-Parallel Graphs	160
4.5	All Planar Graphs	162
4.5.1	The Degree of a Rooted Vertex	162
4.5.2	Singular Expansions	166

4.5.3	Degree Distribution for Planar Graphs	170
4.5.4	Vertices of Degree 1 or 2 in Planar Graphs	174
5	The Shape of Galton-Watson Trees and Pólya Trees	179
5.1	The Continuum Random Tree	180
5.1.1	Depth-First Search of a Rooted Tree	180
5.1.2	Real Trees	181
5.1.3	Galton-Watson Trees and the Continuum Random Tree	183
5.2	The Profile of Galton-Watson Trees	187
5.2.1	The Distribution of the Local Time	190
5.2.2	Weak Convergence of Continuous Stochastic Processes .	192
5.2.3	Combinatorics on the Profile of Galton-Watson Trees .	197
5.2.4	Asymptotic Analysis of the Main Recurrence	198
5.2.5	Finite Dimensional Limiting Distributions	201
5.2.6	Tightness	206
5.2.7	The Height of Galton-Watson Trees	211
5.2.8	Depth-First Search	221
5.3	The Profile of Pólya Trees	226
5.3.1	Combinatorial Setup	227
5.3.2	Asymptotic Analysis of the Main Recurrence	228
5.3.3	Finite Dimensional Limiting Distributions	236
5.3.4	Tightness	240
5.3.5	The Height of Pólya Trees	249
6	The Vertical Profile of Trees	259
6.1	Quadrangulations and Embedded Trees	260
6.2	Profiles of Trees and Random Measures	268
6.2.1	General Profiles	268
6.2.2	Space Embedded Trees and ISE	268
6.2.3	The Distribution of the ISE	273
6.3	Combinatorics on Embedded Trees	276
6.3.1	Embedded Trees with Increments ± 1	276
6.3.2	Embedded Trees with Increments $0, \pm 1$	283
6.3.3	Naturally Embedded Binary Trees	285
6.4	Asymptotics on Embedded Trees	287
6.4.1	Trees with Small Labels	287
6.4.2	The Number of Nodes of Given Label	293
6.4.3	The Number of Nodes of Large Labels	298
6.4.4	Embedded Trees with Increments 0 and ± 1	303
6.4.5	Naturally Embedded Binary Trees	304
6.4.6	Comments on the General Case	304

7	Recursive Trees and Binary Search Trees	309
7.1	Permutations and Trees	310
7.1.1	Permutations and Recursive Trees	311
7.1.2	Permutations and Binary Search Trees	318
7.2	Generating Functions and Basic Statistics	319
7.2.1	Generating Functions for Recursive Trees	320
7.2.2	Generating Functions for Binary Search Trees	321
7.2.3	Generating Functions for Plane Oriented Recursive Trees	323
7.2.4	The Degree Distribution of Recursive Trees	325
7.2.5	The Insertion Depth	334
7.3	The Profile of Recursive Trees	337
7.3.1	The Martingale Method	338
7.3.2	The Moment Method	347
7.3.3	The Contraction Method	351
7.4	The Height of Recursive Trees	352
7.5	Profile and Height of Binary Search Trees	364
7.5.1	The Profile of Binary Search Trees and Related Trees ..	364
7.5.2	The Height of Binary Search Trees and Related Trees ..	373
8	Tries and Digital Search Trees	379
8.1	The Profile of Tries	380
8.1.1	Generating Functions for the Profile	380
8.1.2	The Expected Profile of Tries	383
8.1.3	The Limiting Distribution of the Profile of Tries	393
8.1.4	The Height of Tries	395
8.1.5	Symmetric Tries	396
8.2	The Profile of Digital Search Trees	397
8.2.1	Generating Functions for the Profile	398
8.2.2	The Expected Profile of Digital Search Trees	399
8.2.3	Symmetric Digital Search Trees	410
9	Recursive Algorithms and the Contraction Method	415
9.1	The Number of Comparisons in Quicksort	417
9.2	The L_2 Setting of the Contraction Method	422
9.2.1	A General Type of Recurrence	422
9.2.2	A general L_2 convergence theorem	424
9.2.3	Applications of the L_2 Setting	429
9.3	Limitations of the L_2 Setting and Extensions	433
9.3.1	The Zolotarev metric	434
9.3.2	Degenerate Limit Equations	435
	Appendix	437
	References	443

Index 455