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The Objective Method: Probabilistic Combinatorial Optimization and Local Weak Convergence

David Aldous and J. Michael Steele

1 Introduction

This survey describes a general approach to a class of problems that arise in combinatorial probability and combinatorial optimization. Formally, the method is part of weak convergence theory, but in concrete problems the method has a flavor of its own. A characteristic element of the method is that it often calls for one to introduce a new, infinite, probabilistic *object* whose *local properties* inform us about the *limiting properties* of a sequence of finite problems.

The name *objective method* hopes to underscore the value of shifting ones attention to the new, large random object with fixed distributional properties and way from the sequence of objects with changing distributions. The new object always provides us with some new information on the asymptotic behavior of the original sequence, and, in the happiest cases, the constants associated with the infinite object even permit us to find the elusive limit constants for that sequence.

The Random-Cluster Model

Geoffrey Grimmett

Abstract. The class of random-cluster models is a unification of a variety of stochastic processes of significance for probability and statistical physics, including percolation, Ising, and Potts models; in addition, their study has impact on the theory of certain random combinatorial structures, and of electrical networks. Much (but not all) of the physical theory of Ising/Potts models is best implemented in the context of the random-cluster representation. This systematic summary of random-cluster models includes accounts of the fundamental methods and inequalities, the uniqueness and specification of infinite-volume measures, the existence and nature of the phase transition, and the structure of the subcritical and supercritical phases. The theory for two-dimensional lattices is better developed than for three and more dimensions. There is a rich collection of open problems, including some of substantial significance for the general area of disordered systems, and these are highlighted when encountered. Amongst the major open questions, there is the problem of ascertaining the exact nature of the phase transition for general values of the cluster-weighting factor q, and the problem of proving that the critical random-cluster model in two dimensions, with $1 \le q \le 4$, converges when re-scaled to a stochastic Löwner evolution (SLE). Overall the emphasis is upon the random-cluster model for its own sake, rather than upon its applications to Ising and Potts systems.

1 Introduction

During a classical period, probabilists studied the behaviour of *independent* random variables. The emergent theory is rich, and is linked through theory and application to areas of pure/applied mathematics and to other sciences. It is however unable to answer important questions from a variety of sources concerning large families of *dependent* random variables. Dependence comes in many forms, and one of the targets of modern probability theory has been to derive robust techniques for studying it. The voice of statistical physics has been especially loud in the call for rigour in this general area. In a typical scenario, we are provided with an infinity of random variables, indexed by the vertices of some graph such as the cubic lattice, and which have some dependence structure governed by the geometry of the graph. Thus mathematicians and physicists have had further cause to relate probability and geometry. One

Models of First-Passage Percolation

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Relaxation Times of Markov Chains in Statistical Mechanics and Combinatorial Structures

Fabio Martinelli

Abstract. In Markov chain Monte Carlo theory a particular Markov chain is run for a very long time until its distribution is close enough to the equilibrium measure. In recent years, for models of statistical mechanics and of theoretical computer science, there has been a flourishing of new mathematical ideas and techniques to rigorously control the time it takes for the chain to equilibrate. This has provided a fruitful interaction between the two fields and the purpose of this paper is to provide a comprehensive review of the state of the art.

Random Walks on Finite Groups

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Summary. Markov chains on finite sets are used in a great variety of situations to approximate, understand and sample from their limit distribution. A familiar example is provided by card shuffling methods. From this viewpoint, one is interested in the "mixing time" of the chain, that is, the time at which the chain gives a good approximation of the limit distribution. A remarkable phenomenon known as the cut-off phenomenon asserts that this often happens abruptly so that it really makes sense to talk about "the mixing time". Random walks on finite groups generalize card shuffling models by replacing the symmetric group by other finite groups. One then would like to understand how the structure of a particular class of groups relates to the mixing time of natural random walks on those groups. It turns out that this is an extremely rich problem which is very far to be understood. Techniques from a great variety of different fields – Probability, Algebra, Representation Theory, Functional Analysis, Geometry, Combinatorics – have been used to attack special instances of this problem. This article gives a general overview of this area of research.

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1 Introduction

This article surveys what is known about the convergence of random walks on finite groups, a subject to which Persi Diaconis gives a marvelous introduction in [27]. In the early twentieth century, Markov, Poincaré and Borel discussed the special instance of this problem associated with card shuffling where the underlying group is the symmetric group S_{52}. Two early references are to Emile Borel [15] and K.D. Kosambi and U.V.R. Rao [95]. The early literature focuses mostly on whether or not a given walk is ergodic: for card shuffling, ergodicity means that the deck gets mixed up after many shuffles.