Contents

III Topology and Calculus **1**

27 Limits and Topology **3**
27.1 Introduction 3
27.2 Topologies on Real Vector Spaces 4
27.3 Continuity 14
27.4 Series 21
27.5 Euler's Formula for Polyhedra and Kuratowski's Theorem 30

28 Differentiability **37**
28.1 Introduction 37
28.2 Differentiation 39
28.3 Taylor's Formula 53

29 Inverse and Implicit Functions **59**
29.1 Introduction 59
29.2 The Inverse Function Theorem 60
29.3 The Implicit Function Theorem 64

30 Integration **73**
30.1 Introduction 73
30.2 Partitions and the Integral 74
30.3 Measure and Integrability 81

31 The Fundamental Theorem of Calculus and Fubini's Theorem **87**
31.1 Introduction 87
31.2 The Fundamental Theorem of Calculus 88
31.3 Fubini's Theorem on Iterated Integration 92

32 Vector Fields **97**
32.1 Introduction 97
32.2 Vector Fields 98

33 Fixpoints **105**
33.1 Introduction 105
33.2 Contractions 105

34 Main Theorem of ODEs **113**
34.1 Introduction 113
34.2 Conservative and Time-Dependent Ordinary Differential Equations: The Local Setup 114
34.3 The Fundamental Theorem: Local Version 115
34.4 The Special Case of a Linear ODE 117
34.5 The Fundamental Theorem: Global Version 119

35 Third Advanced Topic **125**
35.1 Introduction 125
35.2 Numerics of ODEs 125
35.3 The Euler Method 129
35.4 Runge-Kutta Methods 131

IV Selected Higher Subjects **137**

36 Categories **139**
36.1 Introduction 139
36.2 What Categories Are 140
36.3 Examples 143
36.4 Functors and Natural Transformations 147
36.5 Limits and Colimits 153
36.6 Adjunction 159

37 Splines **161**
37.1 Introduction 161
37.2 Preliminaries on Simplexes 161
37.3 What are Splines? 164
37.4 Lagrange Interpolation 168
37.5 Bézier Curves 171
37.6 Tensor Product Splines 176
37.7 B-Splines 179

38 Fourier Theory **183**
38.1 Introduction 183
38.2 Spaces of Periodic Functions 185
38.3 Orthogonality 188

38.4 Fourier's Theorem 191
38.5 Restatement in Terms of the Sine and Cosine Functions .. 194
38.6 Finite Fourier Series and Fast Fourier Transform 200
38.7 Fast Fourier Transform (FFT) 204
38.8 The Fourier Transform 209

39 Wavelets 215
39.1 Introduction 215
39.2 The Hilbert Space $L^2(\mathbb{R})$ 217
39.3 Frames and Orthonormal Wavelet Bases 221
39.4 The Fast Haar Wavelet Transform 225

40 Fractals 231
40.1 Introduction 231
40.2 Hausdorff-Metric Spaces 232
40.3 Contractions on Hausdorff-Metric Spaces 236
40.4 Fractal Dimension 242

41 Neural Networks 253
41.1 Introduction 253
41.2 Formal Neurons 254
41.3 Neural Networks 264
41.4 Multi-Layered Perceptrons 269
41.5 The Back-Propagation Algorithm 272

42 Probability Theory 279
42.1 Introduction 279
42.2 Event Spaces and Random Variables 279
42.3 Probability Spaces 283
42.4 Distribution Functions 290
42.5 Expectation and Variance 299
42.6 Independence and the Central Limit Theorem 306
42.7 A Remark on Inferential Statistics 310

43 Lambda Calculus 313
43.1 Introduction 313
43.2 The Lambda Language 314
43.3 Substitution 316
43.4 Alpha-Equivalence 318
43.5 Beta-Reduction 320
43.6 The λ-Calculus as a Programming Language 326

43.7 Recursive Functions 328
43.8 Representation of Partial Recursive Functions 331

A Further Reading **335**

B Bibliography **337**

Index **341**