

Table of Contents

1	Mean-square approximation for stochastic differential equations	1
1.1	Fundamental theorem on the mean-square order of convergence	3
1.1.1	Statement of the theorem	3
1.1.2	Proof of the fundamental theorem	5
1.1.3	The fundamental theorem for equations in the sense of Stratonovich	9
1.1.4	Discussion	10
1.1.5	The explicit Euler method	12
1.1.6	Nonglobally Lipschitz conditions	16
1.2	Methods based on Taylor-type expansion	18
1.2.1	Taylor expansion of solutions of ordinary differential equations	18
1.2.2	Wagner–Platen expansion of solutions of stochastic differential equations	19
1.2.3	Construction of explicit methods	23
1.3	Implicit mean-square methods	29
1.3.1	Construction of drift-implicit methods	29
1.3.2	The balanced method	33
1.3.3	Implicit methods for SDEs with locally Lipschitz vector fields	37
1.3.4	Fully implicit mean-square methods: The main idea	38
1.3.5	Convergence theorem for fully implicit methods	41
1.3.6	General construction of fully implicit methods	43
1.4	Modeling of Ito integrals	45
1.4.1	Ito integrals depending on a single noise and methods of order $3/2$ and 2	45
1.4.2	Modeling Ito integrals by the rectangle and trapezium methods	51
1.4.3	Modeling Ito integrals by the Fourier method	54
1.5	Explicit and implicit methods of order $3/2$ for systems with additive noise	60
1.5.1	Explicit methods based on Taylor-type expansion	60
1.5.2	Implicit methods based on Taylor-type expansion	62

1.5.3	Stiff systems of stochastic differential equations with additive noise. A -stability	66
1.5.4	Runge–Kutta type methods	71
1.5.5	Two-step difference methods	75
1.6	Numerical schemes for equations with colored noise	77
1.6.1	Explicit schemes of orders 2 and $5/2$	78
1.6.2	Runge–Kutta schemes	80
1.6.3	Implicit schemes	81
2	Weak approximation	
	for stochastic differential equations	83
2.1	One-step approximation	87
2.1.1	Properties of remainders and Ito integrals	88
2.1.2	One-step approximations of third order	92
2.1.3	The Taylor expansion of mathematical expectations	98
2.2	The main theorem on convergence of weak approximations and methods of order two	99
2.2.1	The general convergence theorem	100
2.2.2	Runge–Kutta type methods	104
2.2.3	The Talay–Tubaro extrapolation method	105
2.2.4	Implicit method	109
2.3	Weak methods for systems with additive and colored noise	112
2.3.1	Second-order methods	113
2.3.2	Main lemmas for third-order methods	113
2.3.3	Construction of a method of order three	116
2.3.4	Weak schemes for systems with colored noise	121
2.4	Variance reduction	123
2.4.1	The method of important sampling	123
2.4.2	Variance reduction by control variates and combining method	126
2.4.3	Variance reduction for boundary value problems	129
2.5	Application of weak methods to the Monte Carlo computation of Wiener integrals	130
2.5.1	The trapezium, rectangle, and other methods of second order	133
2.5.2	A fourth-order Runge–Kutta method for computing Wiener integrals of functionals of exponential type	135
2.5.3	Explicit Runge–Kutta method of order four for conditional Wiener integrals of exponential-type functionals	137
2.5.4	Theorem on one-step error	140
2.5.5	Implicit Runge–Kutta methods for conditional Wiener integrals of exponential-type functionals	145
2.5.6	Numerical experiments	148
2.6	Random number generators	159

2.6.1	Some uniform random number generators	160
2.6.2	A specific test for SDE integration	163
2.6.3	Generation of Gaussian random numbers	166
2.6.4	Parallel implementation	168
3	Numerical methods for SDEs with small noise	171
3.1	Mean-square approximations and estimation of their errors . .	173
3.1.1	Construction of one-step mean-square approximation . .	173
3.1.2	Theorem on mean-square global estimate	175
3.1.3	Selection of time increment h depending on parameter ε	177
3.1.4	(h, ε) -approach versus (ε, h) -approach	177
3.2	Some concrete mean-square methods for systems with small noise	178
3.2.1	Taylor-type numerical methods	179
3.2.2	Runge–Kutta methods	180
3.2.3	Implicit methods	182
3.2.4	Stratonovich SDEs with small noise	183
3.2.5	Mean-square methods for systems with small additive noise	184
3.3	Numerical tests of mean-square methods	185
3.3.1	Simulation of Lyapunov exponent of a linear system with small noise	185
3.3.2	Stochastic model of a laser	188
3.4	The main theorem on error estimation and general approach to construction of weak methods	190
3.5	Some concrete weak methods	193
3.5.1	Taylor-type methods	193
3.5.2	Runge–Kutta methods	196
3.5.3	Weak methods for systems with small additive noise . .	199
3.6	Expansion of the global error in powers of h and ε	202
3.7	Reduction of the Monte Carlo error	203
3.8	Simulation of the Lyapunov exponent of a linear system with small noise by weak methods	206
4	Stochastic Hamiltonian systems and Langevin-type equations	211
4.1	Preservation of symplectic structure	213
4.2	Mean-square symplectic methods for stochastic Hamiltonian systems	216
4.2.1	General stochastic Hamiltonian systems	216
4.2.2	Explicit methods in the case of separable Hamiltonians	220
4.3	Mean-square symplectic methods for Hamiltonian systems with additive noise	224
4.3.1	The case of a general Hamiltonian	224
4.3.2	The case of separable Hamiltonians	231

4.3.3	The case of Hamiltonian	
	$H(t, p, q) = \frac{1}{2}p^\top M^{-1}p + U(t, q)$	234
4.4	Numerical tests of mean-square symplectic methods	237
4.4.1	Kubo oscillator	237
4.4.2	A model for synchrotron oscillations of particles in storage rings	239
4.4.3	Linear oscillator with additive noise	240
4.5	Liouvillian methods for stochastic systems preserving phase volume	246
4.5.1	Liouvillian methods for partitioned systems with multiplicative noise	248
4.5.2	Liouvillian methods for a volume-preserving system with additive noise	250
4.6	Weak symplectic methods for stochastic Hamiltonian systems	251
4.6.1	Hamiltonian systems with multiplicative noise	251
4.6.2	Hamiltonian systems with additive noise	255
4.6.3	Numerical tests	257
4.7	Quasi-symplectic mean-square methods for Langevin-type equations	261
4.7.1	Langevin equation: Linear damping and additive noise	262
4.7.2	Langevin-type equation: Nonlinear damping and multiplicative noise	270
4.8	Quasi-symplectic weak methods for Langevin-type equations	273
4.8.1	Langevin equation: Linear damping and additive noise	273
4.8.2	Langevin-type equation: Nonlinear damping and multiplicative noise	275
4.8.3	Numerical examples	276
5	Simulation of space and space-time bounded diffusions	283
5.1	Mean-square approximation for autonomous SDEs without drift in a space bounded domain	286
5.1.1	Local approximation of diffusion in a space bounded domain	287
5.1.2	Global algorithm for diffusion in a space bounded domain	291
5.1.3	Simulation of exit point $X_x(\tau_x)$	301
5.2	Systems with drift in a space bounded domain	302
5.3	Space-time Brownian motion	306
5.3.1	Auxiliary knowledge	306
5.3.2	Some distributions for one-dimensional Wiener process	308
5.3.3	Simulation of exit time and exit point of Wiener process from a cube	313
5.3.4	Simulation of exit point of the space-time Brownian motion from a space-time parallelepiped with cubic base	316

5.4	Approximations for SDEs in a space-time bounded domain ..	317
5.4.1	Local mean-square approximation in a space-time bounded domain	318
5.4.2	Global algorithm in a space-time bounded domain	322
5.4.3	Approximation of exit point $(\tau, X(\tau))$	325
5.4.4	Simulation of space-time Brownian motion with drift .	328
5.5	Numerical examples	329
5.6	Mean-square approximation of diffusion with reflection	337
6	Random walks for linear boundary value problems	339
6.1	Algorithms for solving the Dirichlet problem based on time-step control	339
6.1.1	Theorems on one-step approximation	341
6.1.2	Numerical algorithms and convergence theorems	348
6.2	The simplest random walk for the Dirichlet problem for parabolic equations	353
6.2.1	The algorithm of the simplest random walk	353
6.2.2	Convergence theorem	356
6.2.3	Other random walks	359
6.2.4	Numerical tests	364
6.3	Random walks for the elliptic Dirichlet problem	365
6.3.1	The simplest random walk for elliptic equations	366
6.3.2	Other methods for elliptic problems	370
6.3.3	Numerical tests	372
6.4	Specific random walks for elliptic equations and boundary layer	374
6.4.1	Conditional expectation of Ito integrals connected with Wiener process in the ball	376
6.4.2	Specific one-step approximations for elliptic equations .	380
6.4.3	The average number of steps	384
6.4.4	Numerical algorithms and convergence theorems	388
6.5	Methods for elliptic equations with small parameter at higher derivatives	392
6.6	Methods for the Neumann problem for parabolic equations ..	397
6.6.1	One-step approximation for boundary points	399
6.6.2	Convergence theorems	403
7	Probabilistic approach to numerical solution of the Cauchy problem for nonlinear parabolic equations ..	407
7.1	Probabilistic approach to linear parabolic equations	408
7.2	Layer methods for semilinear parabolic equations	415
7.2.1	The construction of layer methods	415
7.2.2	Convergence theorem for a layer method	419
7.2.3	Numerical algorithms	422
7.3	Multi-dimensional case	427

7.3.1	Multidimensional parabolic equation	427
7.3.2	Probabilistic approach to reaction-diffusion systems . . .	429
7.4	Numerical examples	431
7.5	Probabilistic approach to semilinear parabolic equations with small parameter	438
7.5.1	Implicit layer method and its convergence	440
7.5.2	Explicit layer methods	442
7.5.3	Singular case	443
7.5.4	Numerical algorithms based on interpolation	445
7.6	High-order methods for semilinear equation with small constant diffusion and zero advection	446
7.6.1	Two-layer methods	447
7.6.2	Three-layer methods	448
7.7	Numerical tests	451
7.7.1	The Burgers equation with small viscosity	452
7.7.2	The generalized FKPP-equation with a small parameter	455
8	Numerical solution of the nonlinear Dirichlet and Neumann problems based on the probabilistic approach . .	461
8.1	Layer methods for the Dirichlet problem for semilinear parabolic equations	461
8.1.1	Construction of a layer method of first order	463
8.1.2	Convergence theorem	467
8.1.3	A layer method with a simpler approximation near the boundary	468
8.1.4	Numerical algorithms and their convergence	474
8.2	Extension to the multi-dimensional Dirichlet problem	476
8.3	Numerical tests of layer methods for the Dirichlet problems . .	479
8.3.1	The Burgers equation	479
8.3.2	Comparison analysis	482
8.3.3	Quasilinear equation with power law nonlinearities . . .	485
8.4	Layer methods for the Neumann problem for semilinear parabolic equations	488
8.4.1	Construction of layer methods	489
8.4.2	Convergence theorems	493
8.4.3	Numerical algorithms	497
8.4.4	Some other layer methods	499
8.5	Extension to the multi-dimensional Neumann problem	501
8.6	Numerical tests for the Neumann problem	503
8.6.1	Comparison of various layer methods	503
8.6.2	A comparison analysis of layer methods and finite-difference schemes	505

9	Application of stochastic numerics to models with stochastic resonance and to Brownian ratchets 509
9.1	Noise-induced regular oscillations in systems with stochastic resonance 510
9.1.1	Sufficient conditions for regular oscillations 512
9.1.2	Comparison with the approach based on Kramers' theory of diffusion over a potential barrier 517
9.1.3	High-frequency regular oscillations in systems with multiplicative noise 518
9.1.4	Large-amplitude regular oscillations in monostable system 523
9.1.5	Regular oscillations in a system of two coupled oscillators 525
9.2	Noise-induced unidirectional transport 526
9.2.1	Systems with state-dependent diffusion 528
9.2.2	Forced thermal ratchets 533
A	Appendix: Practical guidance to implementation of the stochastic numerical methods 541
A.1	Mean-square methods 541
A.2	Weak methods and the Monte Carlo technique 544
A.3	Algorithms for bounded diffusions 550
A.4	Random walks for linear boundary value problems 558
A.5	Nonlinear PDEs 560
A.6	Miscellaneous 565
	References 571
	Index 587