
Contents

Notation and Conventions	1
<hr/>	
Part Two: Positivity for Vector Bundles	
<hr/>	
Introduction to Part Two	5
6 Ample and Nef Vector Bundles	7
6.1 Classical Theory	7
6.1.A Definition and First Properties	8
6.1.B Cohomological Properties	11
6.1.C Criteria for Amplitude	15
6.1.D Metric Approaches to Positivity of Vector Bundles	18
6.2 \mathbf{Q} -Twisted and Nef Bundles	20
6.2.A Twists by \mathbf{Q} -Divisors	20
6.2.B Nef Bundles	24
6.3 Examples and Constructions	27
6.3.A Normal and Tangent Bundles	27
6.3.B Ample Cotangent Bundles and Hyperbolicity	36
6.3.C Picard Bundles	44
6.3.D The Bundle Associated to a Branched Covering	47
6.3.E Direct Images of Canonical Bundles	51
6.3.F Some Constructions of Positive Vector Bundles	53
6.4 Ample Vector Bundles on Curves	56
6.4.A Review of Semistability	57
6.4.B Semistability and Amplitude	60
Notes	64
7 Geometric Properties of Ample Bundles	65
7.1 Topology	65
7.1.A Sommese's Theorem	65

7.1.B	Theorem of Bloch and Gieseker	68
7.1.C	A Barth-Type Theorem for Branched Coverings	71
7.2	Degeneracy Loci	74
7.2.A	Statements and First Examples	74
7.2.B	Proof of Connectedness of Degeneracy Loci	78
7.2.C	Some Applications	82
7.2.D	Variants and Extensions	87
7.3	Vanishing Theorems	89
7.3.A	Vanishing Theorems of Griffiths and Le Potier	89
7.3.B	Generalizations	95
	Notes	98
8	Numerical Properties of Ample Bundles	101
8.1	Preliminaries from Intersection Theory	101
8.1.A	Chern Classes for \mathbf{Q} -Twisted Bundles	102
8.1.B	Cone Classes	104
8.1.C	Cone Classes for \mathbf{Q} -Twists	110
8.2	Positivity Theorems	111
8.2.A	Positivity of Chern Classes	111
8.2.B	Positivity of Cone Classes	114
8.3	Positive Polynomials for Ample Bundles	117
8.4	Some Applications	125
8.4.A	Positivity of Intersection Products	125
8.4.B	Non-Emptiness of Degeneracy Loci	127
8.4.C	Singularities of Hypersurfaces Along a Curve	129
	Notes	132

Part Three: Multiplier Ideals and Their Applications

	Introduction to Part Three	135
9	Multiplier Ideal Sheaves	139
9.1	Preliminaries	140
9.1.A	\mathbf{Q} -Divisors	140
9.1.B	Normal Crossing Divisors and Log Resolutions	142
9.1.C	The Kawamata–Viehweg Vanishing Theorem	147
9.2	Definition and First Properties	151
9.2.A	Definition of Multiplier Ideals	152
9.2.B	First Properties	158
9.3	Examples and Complements	162
9.3.A	Multiplier Ideals and Multiplicity	162
9.3.B	Invariants Arising from Multiplier Ideals	165
9.3.C	Monomial Ideals	170
9.3.D	Analytic Construction of Multiplier Ideals	176

9.3.E	Adjoint Ideals	179
9.3.F	Multiplier and Jacobian Ideals	181
9.3.G	Multiplier Ideals on Singular Varieties	182
9.4	Vanishing Theorems for Multiplier Ideals	185
9.4.A	Local Vanishing for Multiplier Ideals	186
9.4.B	The Nadel Vanishing Theorem	188
9.4.C	Vanishing on Singular Varieties	191
9.4.D	Nadel’s Theorem in the Analytic Setting	192
9.4.E	Non-Vanishing and Global Generation	193
9.5	Geometric Properties of Multiplier Ideals	195
9.5.A	Restrictions of Multiplier Ideals	195
9.5.B	Subadditivity	201
9.5.C	The Summation Theorem	204
9.5.D	Multiplier Ideals in Families	210
9.5.E	Coverings	213
9.6	Skoda’s Theorem	216
9.6.A	Integral Closure of Ideals	216
9.6.B	Skoda’s Theorem: Statements	221
9.6.C	Skoda’s Theorem: Proofs	226
9.6.D	Variants	228
	Notes	230
10	Some Applications of Multiplier Ideals	233
10.1	Singularities	233
10.1.A	Singularities of Projective Hypersurfaces	233
10.1.B	Singularities of Theta Divisors	235
10.1.C	A Criterion for Separation of Jets of Adjoint Series	238
10.2	Matsusaka’s Theorem	239
10.3	Nakamaye’s Theorem on Base Loci	246
10.4	Global Generation of Adjoint Linear Series	251
10.4.A	Fujita Conjecture and Angehrn–Siu Theorem	252
10.4.B	Loci of Log-Canonical Singularities	254
10.4.C	Proof of the Theorem of Angehrn and Siu	258
10.5	The Effective Nullstellensatz	262
	Notes	267
11	Asymptotic Constructions	269
11.1	Construction of Asymptotic Multiplier Ideals	270
11.1.A	Complete Linear Series	270
11.1.B	Graded Systems of Ideals and Linear Series	276
11.2	Properties of Asymptotic Multiplier Ideals	282
11.2.A	Local Statements	282
11.2.B	Global Results	285
11.2.C	Multiplicativity of Plurigenera	292
11.3	Growth of Graded Families and Symbolic Powers	293

XIV Contents

11.4 Fujita's Approximation Theorem	299
11.4.A Statement and First Consequences	299
11.4.B Proof of Fujita's Theorem.....	305
11.4.C The Dual of the Pseudoeffective Cone	307
11.5 Siu's Theorem on Plurigenera	312
Notes.....	320
References	323
Glossary of Notation	357
Index	363

Contents of Volume I

Notation and Conventions	I: 1
---------------------------------	------

Part One: Ample Line Bundles and Linear Series

Introduction to Part One	I: 5
---------------------------------	------

1 Ample and Nef Line Bundles	I: 7
1.1 Preliminaries: Divisors, Line Bundles, and Linear Series	I: 7
1.1.A Divisors and Line Bundles	I: 8
1.1.B Linear Series	I: 12
1.1.C Intersection Numbers and Numerical Equivalence	I: 15
1.1.D Riemann–Roch	I: 20
1.2 The Classical Theory	I: 24
1.2.A Cohomological Properties	I: 25
1.2.B Numerical Properties	I: 33
1.2.C Metric Characterizations of Amplitude	I: 39
1.3 \mathbf{Q} -Divisors and \mathbf{R} -Divisors	I: 44
1.3.A Definitions for \mathbf{Q} -Divisors	I: 44
1.3.B \mathbf{R} -Divisors and Their Amplitude	I: 48
1.4 Nef Line Bundles and Divisors	I: 50
1.4.A Definitions and Formal Properties	I: 51
1.4.B Kleiman’s Theorem	I: 53
1.4.C Cones	I: 59
1.4.D Fujita’s Vanishing Theorem	I: 65
1.5 Examples and Complements	I: 70
1.5.A Ruled Surfaces	I: 70
1.5.B Products of Curves	I: 73
1.5.C Abelian Varieties	I: 79
1.5.D Other Varieties	I: 80
1.5.E Local Structure of the Nef Cone	I: 82

1.5.F	The Cone Theorem	I: 86
1.6	Inequalities	I: 88
1.6.A	Global Results	I: 88
1.6.B	Mixed Multiplicities	I: 91
1.7	Amplitude for a Mapping	I: 94
1.8	Castelnuovo–Mumford Regularity	I: 98
1.8.A	Definitions, Formal Properties, and Variants	I: 99
1.8.B	Regularity and Complexity	I: 107
1.8.C	Regularity Bounds	I: 110
1.8.D	Syzygies of Algebraic Varieties	I: 115
	Notes	I: 119
2	Linear Series	I: 121
2.1	Asymptotic Theory	I: 121
2.1.A	Basic Definitions	I: 122
2.1.B	Semiample Line Bundles	I: 128
2.1.C	Iitaka Fibration	I: 133
2.2	Big Line Bundles and Divisors	I: 139
2.2.A	Basic Properties of Big Divisors	I: 139
2.2.B	Pseudoeffective and Big Cones	I: 145
2.2.C	Volume of a Big Divisor	I: 148
2.3	Examples and Complements	I: 157
2.3.A	Zariski’s Construction	I: 158
2.3.B	Cutkosky’s Construction	I: 159
2.3.C	Base Loci of Nef and Big Linear Series	I: 164
2.3.D	The Theorem of Campana and Peternell	I: 166
2.3.E	Zariski Decompositions	I: 167
2.4	Graded Linear Series and Families of Ideals	I: 172
2.4.A	Graded Linear Series	I: 172
2.4.B	Graded Families of Ideals	I: 176
	Notes	I: 183
3	Geometric Manifestations of Positivity	I: 185
3.1	The Lefschetz Theorems	I: 185
3.1.A	Topology of Affine Varieties	I: 186
3.1.B	The Theorem on Hyperplane Sections	I: 192
3.1.C	Hard Lefschetz Theorem	I: 199
3.2	Projective Subvarieties of Small Codimension	I: 201
3.2.A	Barth’s Theorem	I: 201
3.2.B	Hartshorne’s Conjectures	I: 204
3.3	Connectedness Theorems	I: 207
3.3.A	Bertini Theorems	I: 207
3.3.B	Theorem of Fulton and Hansen	I: 210
3.3.C	Grothendieck’s Connectedness Theorem	I: 212
3.4	Applications of the Fulton–Hansen Theorem	I: 213
3.4.A	Singularities of Mappings	I: 214

3.4.B	Zak's Theorems	I: 219
3.4.C	Zariski's Problem	I: 227
3.5	Variants and Extensions	I: 231
3.5.A	Homogeneous Varieties	I: 231
3.5.B	Higher Connectivity	I: 233
	Notes	I: 237
4	Vanishing Theorems	I: 239
4.1	Preliminaries	I: 240
4.1.A	Normal Crossings and Resolutions of Singularities	I: 240
4.1.B	Covering Lemmas	I: 242
4.2	Kodaira and Nakano Vanishing Theorems	I: 248
4.3	Vanishing for Big and Nef Line Bundles	I: 252
4.3.A	Statement and Proof of the Theorem	I: 252
4.3.B	Some Applications	I: 257
4.4	Generic Vanishing Theorem	I: 261
	Notes	I: 267
5	Local Positivity	I: 269
5.1	Seshadri Constants	I: 269
5.2	Lower Bounds	I: 278
5.2.A	Background and Statements	I: 278
5.2.B	Multiplicities of Divisors in Families	I: 282
5.2.C	Proof of Theorem 5.2.5	I: 286
5.3	Abelian Varieties	I: 290
5.3.A	Period Lengths and Seshadri Constants	I: 290
5.3.B	Proof of Theorem 5.3.6	I: 297
5.3.C	Complements	I: 301
5.4	Local Positivity Along an Ideal Sheaf	I: 303
5.4.A	Definition and Formal Properties of the s -Invariant	I: 303
5.4.B	Complexity Bounds	I: 308
	Notes	I: 312
<hr/>		
Appendices		
<hr/>		
A	Projective Bundles	I: 315
B	Cohomology and Complexes	I: 317
	B.1 Cohomology	I: 317
	B.2 Complexes	I: 320
	References	I: 325
	Glossary of Notation	I: 359
	Index	I: 365