
Contents

1	Introduction	1
2	Kohn's Proof of the Hypoellipticity of the Hörmander Operators	11
2.1	Vector Fields and Hörmander Condition	11
2.2	Main Results in Hypoellipticity	12
2.3	Kohn's Proof	14
3	Compactness Criteria for the Resolvent of Schrödinger Operators	19
3.1	Introduction	19
3.2	About Witten Laplacians and Schrödinger Operators	20
3.3	Compact Resolvent and Magnetic Bottles	22
4	Global Pseudo-differential Calculus	27
4.1	The Weyl-Hörmander Pseudo-differential Calculus	27
4.2	Basic Properties	29
4.2.1	Composition	29
4.2.2	The Algebra $\cup_{m \in \mathbb{R}} \text{Op } S_{\psi}^m$	30
4.2.3	Equivalence of Quantizations	30
4.2.4	$L^2(\mathbb{R}^d)$ -Continuity	31
4.2.5	Compact Pseudo-differential Operators	31
4.3	Fully Elliptic Operators and Beals Type Characterization	31
4.4	Powers of Positive Elliptic Operators	34
4.5	Comments	37
4.6	Other Types of Pseudo-differential Calculus	38
4.7	A Remark by J.M. Bony About the Geodesic Temperance	39
5	Analysis of Some Fokker-Planck Operator	43
5.1	Introduction	43
5.2	Maximal Accretivity of the Fokker-Planck Operator	43

5.2.1	Accretive Operators	43
5.2.2	Application to the Fokker-Planck Operator	44
5.3	Sufficient Conditions for the Compactness of the Resolvent of the Fokker-Planck Operator	46
5.3.1	Main Result	46
5.3.2	A Metric Adapted to the Fokker-Planck Equation and Weak Ellipticity Assumptions	48
5.3.3	Algebraic Properties of the Fokker-Planck Operator	52
5.3.4	Hypoelliptic Estimates: A Basic Lemma	54
5.3.5	Proof of Theorem 5.8	55
5.4	Necessary Conditions with Respect to the Corresponding Witten Laplacian	58
5.5	Analysis of the Fokker-Planck Quadratic Model	59
5.5.1	Explicit Computation of the Spectrum	60
5.5.2	Improved Estimates for the Quadratic Potential	62
6	Return to Equilibrium for the Fokker-Planck Operator	65
6.1	Abstract Analysis	65
6.2	Applications to the Fokker-Planck Operator	69
6.3	Return to Equilibrium Without Compact Resolvent	70
6.4	On Other Links Between Fokker-Planck Operators and Witten Laplacians	71
6.5	Fokker-Planck Operators and Kinetic Equations	72
7	Hypoellipticity and Nilpotent Groups	73
7.1	Introduction	73
7.2	Nilpotent Lie Algebras	73
7.3	Representation Theory	74
7.4	Rockland's Conjecture	76
7.5	Spectral Properties	77
8	Maximal Hypoellipticity for Polynomial of Vector Fields and Spectral Byproducts	79
8.1	Introduction	79
8.2	Rothschild-Stein Lifting and Towards a General Criterion	80
8.3	Folland's Result	83
8.4	Discussion on Rothschild-Stein and Helffer-Métivier-Nourrigat Results	85
9	On Fokker-Planck Operators and Nilpotent Techniques	89
9.1	Is There a Lie Algebra Approach for the Fokker-Planck Equation?	89
9.2	Maximal Estimates for Some Fokker-Planck Operators	91

10 Maximal Microhypoellipticity for Systems and Applications to Witten Laplacians 97

10.1 Introduction 97

10.2 Microlocal Hypoellipticity and Semi-classical Analysis 99

 10.2.1 Analysis of the Links 99

 10.2.2 Analysis of the Microhypoellipticity for Systems 101

10.3 Around the Proof of Theorem 10.5 103

10.4 Spectral By-products for the Witten Laplacians 106

 10.4.1 Main Statements 106

 10.4.2 Applications for Homogeneous Examples 107

 10.4.3 Applications for Non-homogeneous Examples 110

11 Spectral Properties of the Witten-Laplacians in Connection with Poincaré Inequalities for Laplace Integrals 113

11.1 Laplace Integrals and Associated Laplacians 113

11.2 Links with the Witten Laplacians 114

 11.2.1 On Poincaré and Brascamp-Lieb Inequalities 114

 11.2.2 Links with Spectra of Higher Order Witten Laplacians . . 115

11.3 Some Necessary and Sufficient Conditions for Polyhomogeneous Potentials 117

 11.3.1 Non-negative Polyhomogeneous Potential Near Infinity . 117

 11.3.2 Analysis of the Kernel 119

 11.3.3 Non-positive Polyhomogeneous Potential Near Infinity . 119

11.4 Applications in the Polynomial Case 120

 11.4.1 Main Result 120

 11.4.2 Examples 121

11.5 About the Poincaré Inequality for an Homogeneous Potential . 122

 11.5.1 Necessary Conditions 122

 11.5.2 Sufficient Conditions 124

 11.5.3 The Analytic Case 127

 11.5.4 Homotopy Properties 130

12 Semi-classical Analysis for the Schrödinger Operator: Harmonic Approximation 133

12.1 Introduction 133

12.2 The Case of Dimension 1 133

12.3 Quadratic Models 138

12.4 The Harmonic Approximation, Analysis in Large Dimension . . 139

13 Decay of Eigenfunctions and Application to the Splitting . . 147

13.1 Introduction 147

13.2 Energy Inequalities 147

13.3 The Agmon Distance 148

13.4 Decay of Eigenfunctions for the Schrödinger Operator 149

13.5	Estimates on the Resolvent	151
13.6	WKB Constructions	152
13.7	Upper Bounds for the Splitting	
	Between the Two First Eigenvalues	155
	13.7.1 Rough Estimates	155
	13.7.2 Towards More Precise Estimates	157
	13.7.3 Historical Remarks	157
13.8	Interaction Matrix for the Symmetric Double Well Problem	157
14	Semi-classical Analysis and Witten Laplacians:	
	Morse Inequalities	163
14.1	De Rham Complex	163
14.2	Useful Formulas	164
14.3	Computation of the Witten Laplacian	
	on Functions and 1-Forms	166
14.4	The Morse Inequalities	167
14.5	The Witten Complex	169
14.6	Rough Semi-classical Analysis of the Witten Laplacian	170
15	Semi-classical Analysis and Witten Laplacians:	
	Tunneling Effects	173
15.1	Morse Theory, Agmon Distance and Orientation Complex	173
	15.1.1 Morse Function and Agmon Distance	173
	15.1.2 Generic Conditions on Morse Functions	174
	15.1.3 Orientation Complex	175
15.2	Semi-classical Analysis of the Witten Laplacians	176
	15.2.1 One Well Reference Problems	176
	15.2.2 Improved Decay	177
	15.2.3 An Adapted Basis	178
	15.2.4 WKB Approximation	178
15.3	Semi-classical Analysis of the Witten Complex	179
16	Accurate Asymptotics	
	for the Exponentially Small Eigenvalues of $\Delta_{f,h}^{(0)}$	181
16.1	Assumptions and Labelling of Local Minima	181
16.2	Main Result	183
16.3	Proof of Theorem 16.4 in the Case of Two Local Minima	184
16.4	Towards the General Case	187
17	Application to the Fokker-Planck Equation	189
18	Epilogue	193
	References	195
	Index	205