Contents

1 Introduction 1

2 Background 5
2.1 Algebraic Geometry 5
2.1.1 Affine Varieties 5
2.1.2 Dimension Theory 7
2.1.3 Divisor Class Groups 8
2.1.4 Morphisms 10
2.2 Algebraic Groups 12
2.2.1 Algebraic Groups 12
2.2.2 Root Systems, Weyl Groups and Dynkin Diagrams 15
2.2.3 Tits System and Bruhat Decomposition 18
2.2.4 Representations 19
2.2.5 The Class Group of a Reductive Group 21
2.2.6 Actions, Orbits, Invariants and Quotients 23
2.2.7 Cellular Decompositions of Algebraic Varieties 25
2.3 Semigroups 26
2.3.1 Basic Semigroup Theory 26
2.3.2 Strongly π-regular Semigroups 27
2.3.3 Special Types of Semigroups 30
2.4 Exercises 31
2.4.1 Abstract Semigroups 31

3 Algebraic Monoids 33
3.1 Linear Algebraic Monoids 33
3.2 Normal Monoids 37
3.3 D-monoids 38
3.4 Solvable Monoids 40
3.5 Excercises 40
3.5.1 Linear Algebraic Groups 40
3.5.2 Linear Algebraic Semigroups 41

3.5.3 Irreducible Algebraic Semigroups 42

4 Regularity Conditions 45
4.1 Reductive Monoids 45
4.2 Semigroup Structure of Reductive Monoids 47
4.2.1 The Type Map 47
4.3 Solvable Regular Monoids 48
4.4 Regular Algebraic Monoids 50
4.5 Regularity in Codimension One 52
4.6 Exercises 54
4.6.1 D-monoids 54
4.6.2 Regular and Reductive monoids 54
4.6.3 Regularity in Codimension One 55

5 Classification of Reductive Monoids 57
5.1 The Extension Principle 57
5.2 Vinberg's Approach 61
5.3 Algebraic Monoids as Spherical Varieties 64
5.3.1 Spherical Varieties 64
5.3.2 Rittatore's Approach 66
5.3.3 Type Maps and Colors 68

6 Universal Constructions 73
6.1 Quotients 73
6.2 Class Groups of Reductive Monoids 75
6.3 Flat Monoids 78
6.4 Multilined Closure 86
6.5 Normalization and Representations 89
6.6 Exercises 90
6.6.1 Flat Monoids 90

7 Orbit Structure of Reductive Monoids 91
7.1 The System of Idempotents and the Type Map 91
7.2 The Cross Section Lattice and the Weyl Chamber 93
7.3 $\mathcal{J}$-irreducible Monoids 96
7.4 Explicit Calculations of the Type Map 99
7.4.1 The Type Map for the Adjoint Representations 99
7.4.2 Further Examples of the Type Map 104
7.5 2-reducible Reductive Monoids 107
7.5.1 Reductive Monoids and Type Maps 107
7.5.2 The Type Map of a 2-reducible Monoid 110
7.5.3 Calculating the Type Map Geometrically 114
7.5.4 Monoids with $I_+ = \Delta\backslash\{\alpha\}$ and $I_- = \Delta\backslash\{\beta\}$ 117
7.5.5 Monoids with $I_+ = \phi$ and $I_- = \phi$ 119
7.5.6 ($\mathcal{J}$,σ)-irreducible Monoids Revisited 119

7.6 Type Maps in General . . . 122
7.7 Exercises . . . 124
7.7.1 The Cross Section Lattice . . . 124
7.7.2 Idempotents . . . 125

8 The Analogue of the Bruhat Decomposition . . . 127
8.1 The Renner Monoid R . . . 127
8.2 The Analogue of the Tits System . . . 129
8.3 Row Reduced Echelon Form . . . 132
8.4 The Length Function on R . . . 135
8.5 Order-Preserving Elements of R . . . 137
8.6 The Adherence Order on R . . . 140
8.7 The j-order, R^+ and Pennell's Theorem . . . 143
8.8 The Adherence Order on $M_n(K)$ . . . 147
8.9 Exercises . . . 150

9 Representations and Blocks of Algebraic Monoids . . . 153
9.1 Conjugacy Classes and Adjoint Quotient . . . 153
9.2 Rep(M) according to Doty . . . 156
9.3 The Blocks of $M_n(K)$ when $char(K) = p > 0$ . . . 159
9.4 The Blocks of Solvable Algebraic Monoids . . . 160

10 Monoids of Lie Type . . . 167
10.1 Finite Groups of Lie Type . . . 167
10.1.1 σ Preserves Root Length . . . 167
10.1.2 σ Exchanges Root Length . . . 168
10.2 Endomorphisms of Linear Algebraic Monoids . . . 168
10.3 A Detailed Example . . . 169
10.4 Abstract Monoids of Lie Type . . . 172
10.5 Modular Representations of Finite Reductive Monoids . . . 175
10.6 Exercises . . . 184
10.6.1 Weil Zeta Functions . . . 184
10.6.2 Counting Modular Representations . . . 185

11 Cellular Decomposition of Algebraic Monoids . . . 187
11.1 Monoid Cells . . . 188
11.2 Exercises . . . 192

12 Conjugacy Classes . . . 195
12.1 The Basic Conjugacy Theorem . . . 195
12.2 Some Refinements . . . 197
12.3 Putcha's Decomposition and the Nilpotent Variety . . . 199

13 The Centralizer of a Semisimple Element 205
13.1 Introduction 205
13.2 Main Results 206
13.3 The Structure of R_s and M_s 208
13.4 Examples 209

14 Combinatorics Related to Algebraic Monoids 213
14.1 The Adherence Order on WeW 213
14.2 Shellability and Stanley-Reisner Rings 217
14.3 Distribution of Products in Finite Monoids 217
14.3.1 Properties of $h(a)$ 218
14.3.2 Example 223
14.4 Exercises 226

15 Survey of Related Developments 227
15.1 Complex Representation of Finite Reductive Monoids 227
15.2 Finite Semigroups and Highest Weight Categories 228
15.3 Singularities of G-embeddings 230
15.4 Cohomology of G-embeddings 231
15.5 Horospherical Varieties 232
15.6 Monoids associated with Kac-Moody Groups 232

References 235

Index 243