

---

# Contents

Notation .....	XIII
----------------	------

Introduction .....	1
--------------------	---

---

## Part I Ordinary Differential Equations

---

<b>1 The Analytical Behaviour of Solutions</b> .....	9
1.1 Linear Second-Order Problems Without Turning Points .....	11
1.1.1 Asymptotic Expansions .....	12
1.1.2 The Green's Function and Stability Estimates .....	16
1.1.3 A Priori Estimates for Derivatives and Solution Decomposition .....	21
1.2 Linear Second-Order Turning-Point Problems .....	25
1.3 Quasilinear Problems .....	29
1.4 Linear Higher-Order Problems and Systems .....	35
1.4.1 Asymptotic Expansions for Higher-Order Problems ....	35
1.4.2 A Stability Result .....	36
1.4.3 Systems of Ordinary Differential Equations .....	38
<b>2 Numerical Methods for Second-Order Boundary Value Problems</b> .....	41
2.1 Finite Difference Methods on Equidistant Meshes .....	41
2.1.1 Classical Convergence Theory for Central Differencing .....	41
2.1.2 Upwind Schemes .....	45
2.1.3 The Concept of Uniform Convergence .....	57
2.1.4 Uniformly Convergent Schemes of Higher Order .....	66
2.1.5 Linear Turning-Point Problems .....	68
2.1.6 Some Nonlinear Problems .....	71
2.2 Finite Element Methods on Standard Meshes .....	76
2.2.1 Basic Results for Standard Finite Element Methods ....	76

VIII Contents

2.2.2	Upwind Finite Elements . . . . .	79
2.2.3	Stabilized Higher-Order Methods . . . . .	84
2.2.4	Variational Multiscale and Differentiated Residual Methods . . . . .	95
2.2.5	Uniformly Convergent Finite Element Methods . . . . .	104
2.3	Finite Volume Methods . . . . .	114
2.4	Finite Difference Methods on Layer-adapted Grids . . . . .	116
2.4.1	Graded Meshes . . . . .	119
2.4.2	Piecewise Equidistant Meshes . . . . .	127
2.5	Adaptive Strategies Based on Finite Differences . . . . .	141

---

**Part II Parabolic Initial-Boundary Value Problems in One Space Dimension**

---

<b>1</b>	<b>Introduction . . . . .</b>	<b>155</b>
<b>2</b>	<b>Analytical Behaviour of Solutions . . . . .</b>	<b>159</b>
2.1	Existence, Uniqueness, Comparison Principle . . . . .	159
2.2	Asymptotic Expansions and Bounds on Derivatives . . . . .	161
<b>3</b>	<b>Finite Difference Methods . . . . .</b>	<b>169</b>
3.1	First-Order Problems . . . . .	169
3.1.1	Consistency . . . . .	169
3.1.2	Stability . . . . .	171
3.1.3	Convergence in $L_2$ . . . . .	174
3.2	Convection-Diffusion Problems . . . . .	177
3.2.1	Consistency and Stability . . . . .	178
3.2.2	Convergence . . . . .	182
3.3	Polynomial Schemes . . . . .	183
3.4	Uniformly Convergent Methods . . . . .	187
3.4.1	Exponential Fitting in Space . . . . .	188
3.4.2	Layer-Adapted Tensor-Product Meshes . . . . .	189
3.4.3	Reaction-Diffusion Problems . . . . .	191
<b>4</b>	<b>Finite Element Methods . . . . .</b>	<b>195</b>
4.1	Space-Based Methods . . . . .	196
4.1.1	Polynomial Upwinding . . . . .	197
4.1.2	Uniformly Convergent Schemes . . . . .	199
4.1.3	Local Error Estimates . . . . .	203
4.2	Subcharacteristic-Based Methods . . . . .	205
4.2.1	SDFEM in Space-Time . . . . .	206
4.2.2	Explicit Galerkin Methods . . . . .	211
4.2.3	Eulerian-Lagrangian Methods . . . . .	217

**5 Two Adaptive Methods** . . . . . 223  
 5.1 Streamline Diffusion Methods . . . . . 223  
 5.2 Moving Mesh Methods (*r*-refinement) . . . . . 225

**Part III Elliptic and Parabolic Problems in Several Space Dimensions**

**1 Analytical Behaviour of Solutions** . . . . . 235  
 1.1 Classical and Weak Solutions . . . . . 235  
 1.2 The Reduced Problem . . . . . 238  
 1.3 Asymptotic Expansions and Boundary Layers . . . . . 243  
 1.4 A Priori Estimates and Solution Decomposition . . . . . 247

**2 Finite Difference Methods** . . . . . 259  
 2.1 Finite Difference Methods on Standard Meshes . . . . . 259  
     2.1.1 Exponential Boundary Layers . . . . . 259  
     2.1.2 Parabolic Boundary Layers . . . . . 266  
 2.2 Layer-Adapted Meshes . . . . . 268  
     2.2.1 Exponential Boundary Layers . . . . . 268  
     2.2.2 Parabolic Layers . . . . . 274

**3 Finite Element Methods** . . . . . 277  
 3.1 Inverse-Monotonicity-Preserving Methods Based on Finite  
     Volume Ideas . . . . . 278  
 3.2 Residual-Based Stabilizations . . . . . 302  
     3.2.1 Streamline Diffusion Finite Element Method  
         (SDFEM) . . . . . 302  
     3.2.2 Galerkin Least Squares Finite Element Method  
         (GLSFEM) . . . . . 327  
     3.2.3 Residual-Free Bubbles . . . . . 333  
 3.3 Adding Symmetric Stabilizing Terms . . . . . 338  
     3.3.1 Local Projection Stabilization . . . . . 338  
     3.3.2 Continuous Interior Penalty Stabilization . . . . . 352  
 3.4 The Discontinuous Galerkin Finite Element Method . . . . . 363  
     3.4.1 The Primal Formulation for a Reaction-Diffusion  
         Problem . . . . . 363  
     3.4.2 A First-Order Hyperbolic Problem . . . . . 368  
     3.4.3 dGFEM Error Analysis for Convection-Diffusion  
         Problems . . . . . 371  
 3.5 Uniformly Convergent Methods . . . . . 376  
     3.5.1 Operator-Fitted Methods . . . . . 377  
     3.5.2 Layer-Adapted Meshes . . . . . 381  
 3.6 Adaptive Methods . . . . . 407  
     3.6.1 Adaptive Finite Element Methods for Non-Singularly  
         Perturbed Elliptic Problems: an Introduction . . . . . 407

3.6.2	Robust and Semi-Robust Residual Type Error Estimators	414
3.6.3	A Variant of the DWR Method for Streamline Diffusion	421
<b>4</b>	<b>Time-Dependent Problems</b>	427
4.1	Analytical Behaviour of Solutions	428
4.2	Finite Difference Methods	429
4.3	Finite Element Methods	434

---

**Part IV The Incompressible Navier-Stokes Equations**

---

<b>1</b>	<b>Existence and Uniqueness Results</b>	449
<b>2</b>	<b>Upwind Finite Element Method</b>	453
<b>3</b>	<b>Higher-Order Methods of Streamline Diffusion Type</b>	465
3.1	The Oseen Problem	466
3.2	The Navier-Stokes Problem	476
<b>4</b>	<b>Local Projection Stabilization for Equal-Order Interpolation</b>	485
4.1	Local Projection Stabilization in an Abstract Setting	486
4.2	Convergence Analysis	488
4.2.1	The Special Interpolant	488
4.2.2	Stability	489
4.2.3	Consistency Error	491
4.2.4	A priori Error Estimate	492
4.3	Local Projection onto Coarse-Mesh Spaces	498
4.3.1	Simplices	498
4.3.2	Quadrilaterals and Hexahedra	499
4.4	Schemes Based on Enrichment of Approximation Spaces	501
4.4.1	Simplices	502
4.4.2	Quadrilaterals and Hexahedra	502
4.5	Relationship to Subgrid Modelling	504
4.5.1	Two-Level Approach with Piecewise Linear Elements	505
4.5.2	Enriched Piecewise Linear Elements	507
4.5.3	Spectral Equivalence of the Stabilizing Terms on Simplices	508
<b>5</b>	<b>Local Projection Method for Inf-Sup Stable Elements</b>	511
5.1	Discretization by Inf-Sup Stable Elements	512
5.2	Stability and Consistency	514
5.3	Convergence	516
5.3.1	Methods of Order $r$ in the Case $\sigma > 0$	517
5.3.2	Methods of Order $r$ in the Case $\sigma \geq 0$	522
5.3.3	Methods of Order $r + 1/2$	526

**6 Mass Conservation for Coupled Flow-Transport**

**Problems** ..... 529

6.1 A Model Problem ..... 529

6.2 Continuous and Discrete Mass Conservation ..... 530

6.3 Approximated Incompressible Flows ..... 532

6.4 Mass-Conservative Methods ..... 534

    6.4.1 Higher-Order Flow Approximation ..... 534

    6.4.2 Post-Processing of the Discrete Velocity ..... 536

    6.4.3 Scott-Vogelius Elements ..... 542

**7 Adaptive Error Control** ..... 545

**References** ..... 551

**Index** ..... 599