

# Contents

<b>Preface</b>	VII
<b>Nomenclature</b>	XVII
<b>1. Introduction</b>	1
1.1. The basic equations	1
1.1.1. The molecular dynamics	3
1.1.2. The basic equations	3
1.2. Turbulence models	4
1.2.1. Stochastic models for large-scale turbulence	5
1.2.2. Stochastic models for small-scale turbulence	5
1.2.3. The unification of turbulence models	6
Appendix 1A: Filter operations	6
1A.1. Spatial averages	6
1A.2. Ensemble averages	7
1A.3. The ergodic theorem	7
<b>2. Stochastic variables</b>	9
2.1. PDFs of one variable	9
2.1.1. The need for the usage of probabilistic concepts	9
2.1.2. The definition of PDFs	10
2.1.3. General properties of PDFs	12
2.2. The characterization of PDFs by moments	12
2.2.1. The calculation of moments by PDFs	12

2.2.2. The calculation of PDFs by moments	13
2.2.3. An example: the Gaussian PDF and its moments	15
2.3. PDFs of several variables	16
2.3.1. PDFs	16
2.3.2. Correlations	17
2.3.3. Conditional PDFs	18
2.4. Statistically most-likely PDFs	19
2.4.1. The measurement of uncertainty	20
2.4.2. Statistically most-likely PDFs	21
2.5. Examples for statistically most-likely PDFs	22
2.5.1. Second-order SML PDF: unbounded variables	22
2.5.2. Fourth-order SML PDF: unbounded variables	24
2.5.3. Second-order SML PDF: bounded variables	25
2.6. Examples for other PDFs	26
2.6.1. Gamma and exponential PDFs	27
2.6.2. The beta PDF	28
Appendix 2A: Theta and delta functions	29
2A.1. A theta function for one variable	29
2A.2. A delta function for one variable	29
2A.3. The properties of delta functions	30
2A.4. The extension to the case of several variables	31
<b>3. Stochastic processes</b>	33
3.1. PDF transport equations	33
3.1.1. The Kramers-Moyal equation	33
3.1.2. Markov processes	34
3.1.3. Implications for PDF transport equations	35
3.2. The Fokker-Planck equation	36
3.2.1. The Fokker-Planck equation	36
3.2.2. Transport equations for moments	37
3.2.3. The limiting PDF	38
3.3. An exact solution to the Fokker-Planck equation	40
3.3.1. The equation considered	40
3.3.2. The solution to the Fokker-Planck equation	41
3.3.3. Means, variances and correlations	42

---

3.4. Stochastic equations for realizations	43
3.4.1. Stochastic differential equations	44
3.4.2. The relationship to Fokker-Planck equations	46
3.4.3. Monte Carlo simulation	47
3.5. Stochastic modeling	48
3.5.1. The set of variables considered	48
3.5.2. The coefficients of stochastic equations	48
Appendix 3A: The dynamics of relevant variables	49
3A.1. The problem considered	49
3A.2. The projection operator	50
3A.3. An operator identity	51
3A.4. The dynamics of relevant variables	52
3A.5. The equilibrium dynamics of relevant variables	53
3A.6. Colored Gaussian noise	54
3A.7. White Gaussian noise	56
<b>4. The equations of fluid and thermodynamics</b>	<b>57</b>
4.1. The fluid dynamic variables	57
4.1.1. Means conditioned on the position	57
4.1.2. The conditioned velocity PDF	58
4.1.3. The fluid dynamic variables	59
4.2. From the molecular to fluid dynamics	60
4.2.1. A model for the molecular motion	60
4.2.2. The unclosed fluid dynamic equations	61
4.3. The closure of the fluid dynamic equations	63
4.3.1. The calculation of the deviatoric stress tensor	63
4.3.2. The heat flux calculation	65
4.3.3. Scaling parameters	67
4.3.4. The closure of the velocity and energy equations	68
4.3.5. The resulting basic equations	70
4.4. The equations for multicomponent reacting systems	70
4.4.1. The mass fraction equations	70
4.4.2. The caloric equation of state	72
4.4.3. The thermal equation of state	73
4.4.4. The equations for multicomponent reacting systems	75
4.4.5. Incompressible flows	76
4.4.6. The Boussinesq approximation	77

4.5. Direct numerical simulation	79
4.5.1. The energy cascade	79
4.5.2. The simulation of the energy cascade	82
4.6. Reynolds-averaged Navier-Stokes equations	84
4.6.1. Ensemble-averaged equations	84
4.6.2. The calculation of variances	86
4.6.3. The closure of source terms	86
Appendix 4A: Second- and higher-order RANS equations	87
4A.1. Second-order equations	88
4A.2. Third-order equations	89
4A.3. Fourth-order equations	89
<b>5. Stochastic models for large-scale turbulence</b>	<b>91</b>
5.1. A hierarchy of stochastic velocity models	92
5.1.1. An acceleration model	92
5.1.2. A velocity model	94
5.1.3. A position model	96
5.2. The generalized Langevin model for velocities	97
5.2.1. The generalized Langevin model	97
5.2.2. The implied moment transport equations	98
5.2.3. Specifications of the generalized Langevin model	99
5.3. A hierarchy of Langevin models	101
5.3.1. The extended Langevin model	101
5.3.2. The Langevin model	102
5.3.3. The simplified Langevin model	102
5.4. The Kolmogorov constant	104
5.4.1. $C_0$ for an equilibrium turbulent boundary layer	104
5.4.2. An explanation for the variations of $C_0$	106
5.4.3. Landau's objection to universality	108
5.5. A hierarchy of stochastic models for scalars	109
5.5.1. The scalar dynamics	109
5.5.2. A hierarchy of scalar equations	110
5.5.3. The boundedness of scalars	112
5.5.4. Comparisons with DNS	114
5.6. Compressible reacting flow: velocity models	119
5.6.1. Compressibility effects on velocity fields	119

---

5.6.2. Stochastic velocity models	123
5.6.3. Deterministic velocity models: the $k-\epsilon$ model	125
5.6.4. The inclusion of buoyancy effects	129
5.7. Compressible reacting flow: scalar models	131
5.7.1. Stochastic velocity-scalar models	131
5.7.2. Hybrid methods	133
5.7.3. Assumed-shape PDF methods	136
Appendix 5A: Stochastic models and basic equations	136
5A.1. A nonlinear stochastic model	137
5A.2. The consistency with basic equations	138
5A.3. The determination of model coefficients	139
Appendix 5B: Consistent turbulence models	141
5B.1. The model considered	141
5B.2. Coefficient relations	142
5B.3. The determination of stochastic model coefficients	143
5B.4. A consistent RANS model	143
Appendix 5C: Nonlinear stochastic models	144
5C.1. The limitations of the applicability of linear stochastic equations	144
5C.2. A cubic stochastic model	145
5C.3. Comparisons with other methods	146
5C.4. An application to CBL turbulence simulations	147
<b>6. Stochastic models for small-scale turbulence</b>	153
6.1. The generalization of LES by FDF methods	155
6.1.1. The unclosed LES equations	155
6.1.2. The stochastic model considered	156
6.1.3. The closure of LES equations	158
6.1.4. Hybrid methods	159
6.2. The closure of the equation for filtered velocities	160
6.2.1. The transport equation for the SGS stress tensor	160
6.2.2. The general algebraic expression for the SGS stress tensor	161
6.2.3. Linear and quadratic algebraic SGS stress tensor models	163
6.2.4. Scaling analysis	166
6.2.5. The theoretical calculation of parameters	167
6.2.6. Comparison with DNS data	169

6.3. The closure of the scalar FDF transport equation	170
6.3.1. The scalar-conditioned convective flux	170
6.3.2. The diffusion coefficient	171
6.3.3. The scalar mixing frequency	173
6.4. The closure of LES and FDF equations	175
6.4.1. The closure of LES equations	175
6.4.1. The modeling of the dynamics of SGS fluctuations	176
Appendix 6A: The dynamic eddy length scale calculation	177
Appendix 6B: The scalar-conditioned convective flux	178
Appendix 6C: An assumed-shape FDF method	179
<b>7. The unification of turbulence models</b>	181
7.1. The need for the unification of turbulence models	181
7.1.1. Industrial applications of turbulence models	181
7.1.2. Basic studies by DNS	182
7.2. Unified turbulence models	183
7.2.1. A unified stochastic model	184
7.2.2. A unified model for filtered variables	185
7.3. Some unsolved questions	187
7.3.1. The structure of unified turbulence models	187
7.3.2. The parameters of unified turbulence models	188
<b>8. References</b>	189
<b>9. Author index</b>	201
<b>10. Subject index</b>	205