

Contents

1	Introduction	1
1.1	Economics	1
1.2	Statistics	1
1.3	Mathematics	2
1.4	Summary of results	4
1.5	Applications	6
I	Basic Mathematics	7
2	Totally preordered sets	9
2.1	Introduction	9
2.2	Order relations	9
2.2.1	Basic concepts	9
2.2.2	Completion	14
2.2.3	Representation	16
2.3	Topological concepts	18
2.4	The order topology	20
2.5	Representation	23
2.6	Notes	23
2.6.1	Basic concepts	23
2.6.2	Ordered sets	24
2.6.3	Topology and order topology	24
2.6.4	Ordered topological spaces	24
2.6.5	Lexicographic orders	25
2.6.6	Removing gaps	25
2.6.7	Further results	25
3	Preferences and preference functions	27
3.1	Introduction	27
3.2	Representations and representation theorems	27
3.3	Notes	29

4	Totally preordered product sets	31
4.1	Introduction	31
4.2	Independence assumptions	31
4.3	Order topologies on product sets	33
4.4	Existence of real continuous order homomorphisms	37
4.5	Note	38
5	A subset of a product set	39
5.1	Introduction	39
5.2	Independence	40
5.3	A total preorder on the set S_A	40
5.4	The Thomsen and the Reidemeister conditions . . .	43
5.5	Note	45
	5.5.1 The Reidemeister and Thomsen conditions .	45
6	Mean groupoids	49
6.1	Introduction	49
6.2	Definition of a commutative mean groupoid	49
6.3	Completion of commutative mean groupoid	52
6.4	The Aczél Fuchs theorem	54
6.5	Extension of a commutative mean groupoid	57
6.6	The bisymmetry equation	59
6.7	Notes	60
	6.7.1 History and other results	60
	6.7.2 Classifying commutative mean groupoids . .	61
	6.7.3 Lexicographic "mean groupoids"	61
	6.7.4 Totally ordered mixture spaces	61
	6.7.5 Reducible	62
	6.7.6 Products of mean groupoids	62
	6.7.7 Completion	62
	6.7.8 Measurement of magnitudes	63
	6.7.9 The bisymmetry equation	63
	6.7.10 Counter example (Andrew Gleason, Harvard) ¹	65

7	Products of two sets as a mean groupoid	69
7.1	Introduction	69
7.2	Thomsen's and Reidemeister's conditions	70
7.3	$(S, \succeq) = (X \times Y / \sim)$ as a commutative mean groupoid	73
7.4	$f(x, y) = f_1(x) + f_2(y)$	77
7.5	The functional equation $F(x, y) = g^{-1}(f_1(x) + f_2(y))$	77
7.6	Notes	78
7.6.1	History and further results	78
II	Relations on Function Spaces	81
8	Totally preordered function spaces	83
8.1	Introduction	83
8.2	Notation and definitions	85
8.3	Real order homomorphisms	87
8.4	The function space as a mean groupoid	88
8.5	Minimal independence assumptions	91
8.6	Existence of $F: \mathcal{G} \rightarrow \mathbb{R}$ and $f: \mathcal{G} \times \mathcal{A} \rightarrow \mathbb{R}$	95
8.7	$X = \{1, 2, \dots, n\} (\prod_{i \in X} Y_i, \succeq)$	97
8.8	$Y = \{0, 1\}, (\mathcal{A}, \succeq)$	98
8.9	Y a commutative mean groupoid	98
8.9.1	$(\mathcal{H}, \succeq, \circ)$	102
8.10	Y a commutative mean groupoid with zero	103
8.10.1	$(\mathcal{H}, \succeq, \circ_x, \square_x)$	106
8.11	Related functional equations	107
8.12	Notes	108
9	Relations on function spaces	113
9.1	Introduction	113
9.2	Existence of $F: \mathcal{G} \rightarrow \mathbb{R}, f: \mathcal{G} \times \mathcal{A} \rightarrow \mathbb{R}$	113
9.3	Existence of $F: \mathcal{G} \times \mathcal{H} \rightarrow \mathbb{R}, f: \mathcal{G} \times \mathcal{H} \times \mathcal{A} \rightarrow \mathbb{R}$	115
9.3.1	$((X, \mathcal{A}), Y, \mathcal{G}, \mathcal{P})$ Existence of $F: \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{R}, f: \mathcal{G} \times \mathcal{G} \times \mathcal{A} \rightarrow \mathbb{R}$	116
9.4	$X = \{1, 2, \dots, n\} (\prod_{i \in X} Y_i, \prod_{i \in X} Z_i, \mathcal{P})$	117
9.5	$Y = Z = \{0, 1\}, (X, \mathcal{A}, \mathcal{P})$	118
9.6	Minimal independence assumptions	119
9.7	$(Y_x, Q_x)_{x \in X}$	120
9.7.1	$(X, Y, \mathcal{G}, Q, (Q_x)_{x \in X})$	121

9.7.2 $((\mathcal{G}, \sim, \circ), (Y_x, \mathcal{P}_x)_{x \in X}) =$
 $((\mathcal{G}, \sim, \circ), (Y_x \times Y_x) / \sim_x, \succeq_x, \circ_x) \dots\dots\dots 123$

9.7.3 $(\mathcal{G}, \mathcal{P}), (Y_x / \sim_x, \succeq_x, \circ_x) \dots\dots\dots 124$

9.7.4 $(\mathcal{G}, \mathcal{P}), (Y_x / \sim_x, \succeq_x, \circ_x, \square_x) \dots\dots\dots 126$

9.7.5 $((\mathcal{G}, \mathcal{P}), (Y_x, \mathcal{P}_x)_{x \in X}) =$
 $((\mathcal{G} \times \mathcal{G}, \sim, \circ, \square), (Y \times Y) / \sim_x, \succeq_x, \circ_x, \square_x) \dots\dots\dots 126$

9.8 Notes $\dots\dots\dots 128$

III Relations on Measures 131

10 Relations on sets of probability measures 133

10.1 Introduction $\dots\dots\dots 133$

10.2 Definitions and mathematics $\dots\dots\dots 133$

10.3 Existence of a Bernoulli function $\dots\dots\dots 135$

10.4 von Neumann Morgenstern preferences $\dots\dots\dots 136$

 10.4.1 The finite case $\dots\dots\dots 137$

 10.4.2 The general case $\dots\dots\dots 138$

 10.4.3 Special cases $\dots\dots\dots 140$

10.5 Notes $\dots\dots\dots 144$

IV Integral Representations 147

11 A general integral representation by Birgit Grodal 149

11.1 Introduction $\dots\dots\dots 149$

11.2 Existence of $u : X \times Y \rightarrow \mathbb{R}$ with
 $f(g, A) = \int_A u(x, g(x)) d\mu \dots\dots\dots 152$

11.3 Continuity and boundedness of $u \dots\dots\dots 159$

11.4 Existence of $u : X \times Y \rightarrow \mathbb{R}$ when \mathcal{G} is a set of measurable selections. $\dots\dots\dots 164$

11.5 Notes $\dots\dots\dots 165$

12 Special integral representations by Birgit Grodal 169

12.1 Introduction $\dots\dots\dots 169$

12.2 $f(g, A) = \int_A \beta(x, \bar{u}(g(x))) d\mu \dots\dots\dots 170$

12.3 $f(g, A) = \int_A \bar{u}(g(x)) \alpha(x) d\mu \dots\dots\dots 174$

12.4 $f(g, A) = \int_A \bar{u}(t) e^{-\delta t} d\lambda \dots\dots\dots 177$

12.5 Notes 183

V Decompositions and Uncertainty 187

13 Decompositions. Uncertainty 189

13.1 Introduction 189

13.2 von Neumann Morgenstern preferences 190

13.3 Function spaces 192

 13.3.1 $Y = Z = \{0, 1\}$. Subjective probabilities
 and uncertainty 192

 13.3.2 $X = \{1, 2, \dots, n\} (\prod_{i \in X} Y_i, \mathcal{P})$ 193

 13.3.3 Y and X general 194

13.4 Historical notes 195

 13.4.1 Knight 195

 13.4.2 Keynes 195

 13.4.3 von Neumann Morgenstern 196

 13.4.4 Savage 197

 13.4.5 Aumann 197

 13.4.6 Friedman 198

 13.4.7 Bewley 198

13.5 Conclusion 199

14 Uncertainty on products 201

14.1 Introduction 201

14.2 One level uncertainty on factors and products 203

 14.2.1 $Y = Z = \{0, 1\}, X = X_1 \times X_2$ 203

 14.2.2 $Y = Z = \{0, 1\}, (X, \mathcal{A}_i)_{i \in I}$ 206

 14.2.3 Y and Z general 213

14.3 Two level uncertainty 215

14.4 Conclusions 220

14.5 Note 220

15 Conditional uncertainty 221

15.1 Introduction 221

15.2 Relations on function spaces 221

15.3 One probability-uncertainty measure 222

15.4 Several probability-uncertainty measures 223

 15.4.1 $Y = Z = \{0, 1\}$ 223

 15.4.2 Y and Z general 223

15.5	Two level uncertainty	223
15.6	Conclusion	223
VI	Applications	225
16	Production, utility, preference	227
16.1	Introduction	227
16.2	Production functions	227
16.3	Additive preference functions	228
16.4	Additive utility functions	228
16.5	Notes	229
17	Preferences over time	231
17.1	Introduction	231
17.2	$((T, \mathcal{A}), Y, Z, \mathcal{G}, \mathcal{H}, \mathcal{P})$	
	Existence of $f : \mathcal{G} \times \mathcal{H} \times \mathcal{A} \rightarrow \mathbb{R}$	232
	17.2.1 Y general	235
17.3	Existence and decomposition of	
	$f : \mathcal{G} \times \mathcal{H} \times \mathcal{G} \times \mathcal{H} \times \mathcal{A} \rightarrow \mathbb{R}$	237
17.4	Notes	240
18	A foundation for statistics	243
18.1	Introduction and historical background	243
18.2	Basic concepts	244
18.3	Uncertainty about the parameter space.	247
18.4	Robust Bayesian inference	247
18.5	Requirements for a foundation of statistics.	247
18.6	A foundation of statistics	248
18.7	Notes	252
	References	253
	Index	269