

Contents

Part I. Preliminaries

1. Colouring Preliminaries	3
1.1 The Basic Definitions	3
1.2 Some Classical Results	5
1.3 Fundamental Open Problems	7
1.4 A Point of View	9
1.5 A Useful Technical Lemma	10
1.6 Constrained Colourings and the List Chromatic Number	11
1.7 Intelligent Greedy Colouring	12
Exercises	13
2. Probabilistic Preliminaries	15
2.1 Finite Probability Spaces	15
2.2 Random Variables and Their Expectations	17
2.3 One Last Definition	19
2.4 The Method of Deferred Decisions	20
Exercises	21

Part II. Basic Probabilistic Tools

3. The First Moment Method	27
3.1 2-Colouring Hypergraphs	28
3.2 Triangle-Free Graphs with High Chromatic Number	29
3.3 Bounding the List Chromatic Number as a Functione of the Colouring Number	31
3.3.1 An Open Problem	33
3.4 The Cochromatic Number	34
Exercises	36

4. The Lovász Local Lemma	39
4.1 Constrained Colourings and the List Chromatic Number	41
Exercises	42
5. The Chernoff Bound	43
5.1 Hajós’s Conjecture	44
Exercises	46

Part III. Vertex Partitions

6. Hadwiger’s Conjecture	49
6.1 Step 1: Finding a Dense Subgraph	50
6.2 Step 2: Finding a Split Minor	50
6.3 Step 3: Finding the Minor	52
Exercises	53
7. A First Glimpse of Total Colouring	55
8. The Strong Chromatic Number	61
Exercises	65
9. Total Colouring Revisited	67
9.1 The Idea	67
9.2 Some Details	70
9.3 The Main Proof	74
Exercises	75

Part IV. A Naive Colouring Procedure

10. Talagrand’s Inequality and Colouring Sparse Graphs	79
10.1 Talagrand’s Inequality	79
10.2 Colouring Triangle-Free Graphs	83
10.3 Colouring Sparse Graphs	86
10.4 Strong Edge Colourings	87
Exercises	89
11. Azuma’s Inequality and a Strengthening of Brooks’ Theorem	91
11.1 Azuma’s Inequality	91
11.2 A Strengthening of Brooks’ Theorem	94
11.3 The Probabilistic Analysis	98

11.4 Constructing the Decomposition 100
 Exercises 103

Part V. An Iterative Approach

12. Graphs with Girth at Least Five 107
 12.1 Introduction 107
 12.2 A Wasteful Colouring Procedure 109
 12.2.1 The Heart of The Procedure 109
 12.2.2 The Finishing Blow 111
 12.3 The Main Steps of the Proof 112
 12.4 Most of the Details 115
 12.5 The Concentration Details 120
 Exercises 123

13. Triangle-Free Graphs 125
 13.1 An Outline 126
 13.1.1 A Modified Procedure 126
 13.1.2 Fluctuating Probabilities 128
 13.1.3 A Technical Fiddle 130
 13.1.4 A Complication 131
 13.2 The Procedure 131
 13.2.1 Dealing with Large Probabilities 131
 13.2.2 The Main Procedure 132
 13.2.3 The Final Step 132
 13.2.4 The Parameters 133
 13.3 Expectation and Concentration 136
 Exercises 138

14. The List Colouring Conjecture 139
 14.1 A Proof Sketch 140
 14.1.1 Preliminaries 140
 14.1.2 The Local Structure 140
 14.1.3 Rates of Change 141
 14.1.4 The Preprocessing Step 142
 14.2 Choosing $Reserve_e$ 144
 14.3 The Expected Value Details 145
 14.4 The Concentration Details 149
 14.5 The Wrapup 151
 14.6 Linear Hypergraphs 152
 Exercises 153

Part VI. A Structural Decomposition

15. The Structural Decomposition	157
15.1 Preliminary Remarks	157
15.2 The Decomposition	157
15.3 Partitioning the Dense Sets	160
15.4 Graphs with χ Near Δ	165
15.4.1 Generalizing Brooks' Theorem	165
15.4.2 Blowing Up a Vertex	166
Exercises	167
16. ω, Δ and χ	169
16.1 The Modified Colouring Procedure	171
16.2 An Extension of Talagrand's Inequality	172
16.3 Strongly Non-Adjacent Vertices	173
16.4 Many Repeated Colours	175
16.5 The Proof of Theorem 16.5	179
16.6 Proving the Harder Theorems	181
16.7 Two Proofs	182
Exercises	184
17. Near Optimal Total Colouring I: Sparse Graphs	185
17.1 Introduction	185
17.2 The Procedure	187
17.3 The Analysis of the Procedure	188
17.4 The Final Phase	191
18. Near Optimal Total Colouring II: General Graphs	195
18.1 Introduction	195
18.2 Phase I: An Initial Colouring	198
18.2.1 Ornerly Sets	198
18.2.2 The Output of Phase I	200
18.2.3 A Proof Sketch	201
18.3 Phase II: Colouring the Dense Sets	206
18.3.1 \mathcal{Y}_i is Non-Empty	207
18.3.2 Our Distribution is Nearly Uniform	208
18.3.3 Completing the Proof	209
18.4 Phase III: The Temporary Colours	210
18.4.1 Step 1: The Kernels of the Ornerly Sets	211
18.4.2 Step 2: The Remaining Temporary Colours	215
18.5 Phase IV – Finishing the Sparse Vertices	216
18.6 The Ornerly Set Lemmas	217

Part VII. Sharpening our Tools

19. Generalizations of the Local Lemma 221

 19.1 Non-Uniform Hypergraph Colouring 222

 19.2 More Frugal Colouring 224

 19.2.1 Acyclic Edge Colouring 225

 19.3 Proofs 226

 19.4 The Lopsided Local Lemma 228

 Exercises 229

20. A Closer Look at Talagrand’s Inequality 231

 20.1 The Original Inequality 231

 20.2 More Versions 234

 Exercises 236

Part VIII. Colour Assignment via Fractional Colouring

21. Finding Fractional Colourings and Large Stable Sets 239

 21.1 Fractional Colouring 239

 21.2 Finding Large Stable Sets in Triangle-Free Graphs 242

 21.3 Fractionally, $\chi \leq \frac{\omega + \Delta + 1}{2}$ 244

 Exercises 246

22. Hard-Core Distributions on Matchings 247

 22.1 Hard-Core Distributions 247

 22.2 Hard-Core Distributions from Fractional Colourings 249

 22.3 The Mating Map 252

 22.4 An Independence Result 254

 22.5 More Independence Results 260

23. The Asymptotics of Edge Colouring Multigraphs 265

 23.1 Assigning the Colours 265

 23.1.1 Hard-Core Distributions
 and Approximate Independence 266

 23.2 The Chromatic Index 267

 23.3 The List Chromatic Index 270

 23.3.1 Analyzing an Iteration 272

 23.3.2 Analyzing a Different Procedure 274

 23.3.3 One More Tool 277

 23.4 Comparing the Procedures 279

 23.4.1 Proving Lemma 23.9 282

Part IX. Algorithmic Aspects

24. The Method of Conditional Expectations	287
24.1 The Basic Ideas	287
24.2 An Algorithm	288
24.3 Generalized Tic-Tac-Toe	289
24.4 Proof of Lemma 24.3	291
25. Algorithmic Aspects of the Local Lemma	295
25.1 The Algorithm	296
25.1.1 The Basics	296
25.1.2 Further Details	299
25.2 A Different Approach	300
25.3 Applicability of the Technique	301
25.3.1 Further Extensions	303
25.4 Extending the Approach	304
25.4.1 3-Uniform Hypergraphs	305
25.4.2 k -Uniform Hypergraphs with $k \geq 4$	308
25.4.3 The General Technique	310
Exercises	312
References	314
Index	323