

# Contents

<b>Preface</b>	<b>vii</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Scope of this Monograph . . . . .	2
1.2 Useful Background for this Presentation . . . . .	3
1.3 Overview . . . . .	4
<b>2 Finite Element Formulations in Nonlinear Solid Mechanics</b>	<b>7</b>
2.1 Initial/Boundary Value Problems in the Kinematically Linear Regime . . . . .	8
2.1.1 Strong Form of the IBVP . . . . .	9
2.1.2 Weak Form of the IBVP . . . . .	15
2.2 The IBVP in the Finite Strain Case . . . . .	17
2.2.1 Notation and Problem Formulation . . . . .	17
2.2.2 Finite Strain Kinematics . . . . .	18
2.2.3 Stress Definitions Appropriate for Large Deformations	24
2.2.4 Frame Indifference . . . . .	27
2.2.5 The Strong Form in Finite Strains . . . . .	31
2.2.6 The Weak Form in Finite Strains . . . . .	39
2.3 Finite Element Discretization . . . . .	41
2.3.1 Discretized Weak Form; Generation of Discrete Nonlinear Equations . . . . .	43
2.3.2 Discrete Nonlinear Equations for the Kinematically Linear Case . . . . .	46

<b>2.4</b>	<b>Solution Strategies for Spatially Discrete Systems . . . . .</b>	<b>48</b>
2.4.1	Quasistatics and Incremental Load Methods . . . . .	48
2.4.2	Dynamics and Global Time Stepping Procedures . . . . .	50
2.4.3	Local (Constitutive) Time Stepping Procedures . . . . .	54
2.4.4	Nonlinear Equation Solving . . . . .	56
2.4.5	Consistent Algorithmic Linearization of Material Response . . . . .	61
<b>3</b>	<b>The Kinematically Linear Contact Problem</b>	<b>69</b>
3.1	Strong Forms in Linearized Frictionless Contact . . . . .	70
3.1.1	The Signorini Problem: Contact with a Rigid Obstacle	70
3.1.2	The Two Body Contact Problem . . . . .	75
3.2	Weak Statements of the Contact Problem . . . . .	79
3.2.1	Variational Inequalities . . . . .	81
3.2.2	The Quasistatic Elastic Case: Contact as a Problem of Constrained Optimization . . . . .	83
3.3	Methods of Constraint Enforcement . . . . .	85
3.3.1	Classical Lagrange Multiplier Methods . . . . .	85
3.3.2	Penalty Methods . . . . .	89
3.3.3	Augmented Lagrangian Methods . . . . .	91
3.4	Inclusion of Friction into the Problem Description . . . . .	94
3.4.1	Friction Kinematics and Traction Measures . . . . .	94
3.4.2	Unregularized Coulomb Friction Laws . . . . .	96
3.4.3	Regularization of Friction . . . . .	98
3.4.4	Variational Statements Including Friction . . . . .	101
3.4.5	Nonlocal Frictional Descriptions . . . . .	106
<b>4</b>	<b>Continuum Mechanics of Large Deformation Contact</b>	<b>109</b>
4.1	Two Body Contact Problem Definition . . . . .	110
4.1.1	Local Momentum Balances . . . . .	111
4.1.2	Initial and Boundary Conditions . . . . .	112
4.2	Contact Constraints in Large Deformations . . . . .	113
4.2.1	The Gap Function as Defined by Closest Point Projection . . . . .	113
4.2.2	Frictional Kinematics on Interfaces . . . . .	116
4.2.3	Frame Indifference of Contact Rate Variables . . . . .	121
4.2.4	Coulomb Friction in Large Sliding . . . . .	129
4.3	Summary: Strong Form of the Large Deformation Contact Problem . . . . .	134
4.4	Virtual Work Expressions Incorporating Contact . . . . .	137
4.4.1	Contact Virtual Work: The Contact Integral . . . . .	139
4.4.2	Linearization of Contact Virtual Work . . . . .	141
4.4.3	Summary: Weak Form of the Large Deformation Contact Problem . . . . .	144

<b>5 Finite Element Implementation of Contact Interaction</b>	<b>145</b>
5.1 Finite Dimensional Representation of Contact Interaction . . . . .	147
5.1.1 Contact Surface Discretization . . . . .	147
5.1.2 Numerical Integration of the Contact Integral . . . . .	148
5.1.3 Contact Detection (Searching) . . . . .	152
5.2 Time Discretization . . . . .	158
5.2.1 Global time integration schemes . . . . .	158
5.2.2 Temporally Discrete Frictional Laws for the Penalty Regularized Case . . . . .	159
5.3 Contact Stiffness and Residual: Penalty Regularized Case . . . . .	162
5.3.1 Three dimensional matrix expressions . . . . .	162
5.3.2 Two dimensional matrix expressions . . . . .	166
5.4 Augmented Lagrangian Constraint Enforcement Algorithms . . . . .	169
5.4.1 Uzawa's Method (Method of Multipliers) . . . . .	170
5.4.2 Algorithmic Symmetrization Using Augmented Lagrangians . . . . .	174
5.4.3 Augmented Lagrangian Discrete Force and Stiffness Expressions . . . . .	178
5.5 Numerical Examples . . . . .	180
5.5.1 General Demonstrations of the Computational Framework . . . . .	180
5.5.2 Demonstrations of Augmented Lagrangian Algorithmic Performance . . . . .	196
<b>6 Tribological Complexity in Interface Constitutive Models</b>	<b>211</b>
6.1 Rate and State Dependent Friction . . . . .	212
6.1.1 Motivation . . . . .	213
6.1.2 One Dimensional Model Development . . . . .	215
6.1.3 Model Incorporation into Convective Slip Advection Frame . . . . .	220
6.1.4 Local Time Stepping Algorithm . . . . .	222
6.1.5 Contact Force Vector and Stiffness Matrix . . . . .	226
6.1.6 Numerical Examples . . . . .	227
6.2 Thermomechanically Coupled Friction on Interfaces . . . . .	238
6.2.1 Motivation . . . . .	239
6.2.2 Thermally Coupled Problem Definition . . . . .	241
6.2.3 A Thermodynamically Consistent Friction Model . . . . .	244
6.2.4 Variational Principle and Finite Element Implementation . . . . .	255
6.2.5 Numerical Examples . . . . .	269
6.3 Thermodynamical Algorithmic Consistency . . . . .	279
6.3.1 Constitutive Framework for Bulk Continua . . . . .	280
6.3.2 Thermomechanical Interface Model Framework . . . . .	283
6.3.3 <i>A Priori</i> Stability Estimates for Dynamic Frictional Contact . . . . .	286

6.3.4	A New Partitioned Scheme for Thermomechanical Contact . . . . .	289
6.3.5	Algorithmic Treatment of Contact Conditions According to the Adiabatic Split . . . . .	291
<b>7</b>	<b>Energy-Momentum Approaches to Impact Mechanics</b>	<b>295</b>
7.1	Energy Stability of Traditional Schemes . . . . .	297
7.1.1	A Model System . . . . .	297
7.1.2	The Concept of Energy Stability . . . . .	299
7.1.3	Influence of Contact Constraints on System Energy .	300
7.2	Energy-Momentum Methods for Elastodynamics . . . . .	304
7.2.1	Conservation Laws . . . . .	305
7.2.2	Conservative Discretization Schemes . . . . .	309
7.3	Energy-Momentum Algorithmic Treatment of Frictionless Impact . . . . .	312
7.3.1	Discrete Contact Constraints . . . . .	313
7.3.2	Spatial Discretization and Implementation . . . . .	316
7.3.3	Numerical Examples . . . . .	318
7.4	Introduction of Frictional and Bulk Dissipation: Energy Consistency . . . . .	325
7.4.1	Coulomb Friction Model Formulation . . . . .	325
7.4.2	Local Split of the Coulomb Model . . . . .	331
7.4.3	Algorithmic Formulation . . . . .	332
7.4.4	Energy Consistent Treatment of Bulk Inelasticity .	338
7.4.5	Numerical Examples With Friction and Inelasticity .	339
7.5	EM Algorithms Involving a Discontinuous Velocity Update .	347
7.5.1	Temporally Discontinuous Velocity Update . . . . .	348
7.5.2	Reexamination of Conservation Conditions . . . . .	350
7.5.3	Contact Constraints . . . . .	355
7.5.4	Summary of the Algorithm . . . . .	357
7.5.5	Numerical Examples . . . . .	357
<b>8</b>	<b>Emerging Paradigms for Contact Surface Discretization</b>	<b>369</b>
8.1	Contact Smoothing . . . . .	371
8.1.1	An Alternative Variational Framework . . . . .	372
8.1.2	Smoothing Strategies in Two Dimensions . . . . .	374
8.1.3	Smoothing Strategies in Three Dimensions . . . . .	382
8.1.4	Numerical Examples . . . . .	390
8.2	Mortar-Finite Element Methods for Contact Description .	404
8.2.1	Tied Contact and the Role of Mortar Formulations in Convergence . . . . .	404
8.2.2	A Mortar-Finite Element Formulation of Frictional Contact . . . . .	416
8.2.3	Numerical Examples of Mortar Treatment of Frictional Contact . . . . .	425

<b>References</b>	<b>435</b>
<b>Index</b>	<b>451</b>