

Table of Contents

1	Homology 3–Spheres	1
1.1	Integral Homology 3–Spheres	1
1.1.1	Homotopy 3–Spheres	1
1.1.2	Poincaré Homology Sphere	1
1.1.3	Brieskorn Homology Spheres	2
1.1.4	Seifert Fibered Homology Spheres	2
1.1.5	Dehn Surgery on Knots	4
1.1.6	Surgery on Links	7
1.1.7	Connected Sums and Splicing	9
1.1.8	Splice Decomposition	10
1.1.9	Plumbing	12
1.1.10	Links of Singularities	16
1.1.11	Mutations	18
1.1.12	Branched Covers	19
1.1.13	Heegaard Splittings of Homology Spheres	25
1.2	Rational Homology Spheres	26
1.2.1	Spherical Space Forms	26
1.2.2	Dehn Surgery	29
1.2.3	Seifert Fibered Manifolds	31
1.2.4	Links of Singularities	32
1.2.5	Branched Covers	33
2	Rokhlin Invariant	35
2.1	The Rokhlin Theorem	35
2.2	Definition of the Rokhlin Invariant	38
2.3	Properties of the Rokhlin Invariant	39
2.3.1	Surgery Formula for the Rokhlin Invariant	39
2.3.2	Surgery on Algebraically Split Links	41
2.3.3	Splicing and Mutation	42
2.3.4	Rokhlin Invariant of Branched Coverings	43
2.3.5	Birman–Craggs Homomorphisms	43
2.3.6	Homology Cobordism Invariance	44
2.4	Seifert Fibered and Graph Homology Spheres	45
2.4.1	The Algorithm	45
2.4.2	The Formula	48

3	Casson Invariant	51
3.1	Definition of the Casson Invariant	51
3.2	Construction of the Casson Invariant	54
3.2.1	$SU(2)$ –Representation Spaces	54
3.2.2	The Intersection Theory	55
3.2.3	Orientations	57
3.2.4	Independence of Heegaard Splitting	58
3.2.5	Casson Invariant for Knots and Property (1)	58
3.2.6	The Difference Cycle	60
3.2.7	Casson Invariant for Boundary Links and Property (2)	61
3.2.8	Casson Invariant of a Trefoil and Property (0)	62
3.3	Comments and Ramifications	63
3.3.1	Pillowcase	63
3.3.2	Perturbations	66
3.3.3	The Connected Sum Formula	67
3.3.4	The Integrality of $\lambda(\Sigma)$	68
3.3.5	Casson Invariant of Algebraically Split Links	68
3.4	Properties of the Casson Invariant	69
3.4.1	Splicing Additivity	69
3.4.2	Mutation Invariance	70
3.4.3	Casson Invariant of Branched Coverings	70
3.4.4	Casson Invariant of Fibered Knots	73
3.4.5	Finite Type Invariants	73
3.4.6	Further Properties of the Casson Invariant	74
3.5	Seifert Fibered and Graph Homology Spheres	75
3.5.1	Casson Invariant of $\Sigma(p, q, r)$	75
3.5.2	Casson Invariant of $\Sigma(a_1, \dots, a_n)$	78
3.5.3	The Neumann–Wahl Conjecture	82
3.6	Applications of the Casson Invariant	83
3.6.1	Triangulating Topological 4–Manifolds	83
3.6.2	Amphicheiral Homology Spheres	84
3.6.3	Property P for Knots	85
4	Invariants of Walker and Lescop	87
4.1	Definition of the Walker Invariant	87
4.2	Construction of the Walker Invariant	88
4.2.1	$SU(2)$ –Representation Varieties	89
4.2.2	The Intersection Theory	89
4.2.3	The Surgery Formula	92
4.2.4	Combinatorial Definition of the Walker Invariant	92
4.3	The Lescop Invariant	93
4.4	Properties of the Walker and Lescop Invariants	95
4.4.1	The Gluing Formula	96
4.4.2	Branched Covers	96
4.4.3	Seifert Fibered Manifolds	98
4.5	Casson Type Invariants from Other Lie Groups	98

5	Casson Invariant and Gauge Theory	99
5.1	Gauge Theory in Dimension 3	99
5.2	Chern-Simons Function	101
5.3	The Casson Invariant from Gauge Theory	103
5.3.1	Morse Theory and Euler Characteristic	103
5.3.2	Critical Points of cs and Spectral Flow	103
5.3.3	Non-degenerate Case	105
5.3.4	Perturbations	106
5.3.5	Morse-type Perturbations	108
5.3.6	Casson Invariant and Seiberg-Witten Equations	109
5.4	Casson-type Invariants of Knots	110
5.4.1	Representation Varieties of Knot Groups	110
5.4.2	The Invariants	112
5.5	Equivariant Casson Invariant	113
5.5.1	Equivariant Gauge Theory	113
5.5.2	Definition of the Invariants	114
5.5.3	Equivariant Casson and Knot Signatures	115
5.5.4	Applications	116
5.6	The $SU(3)$ Casson Invariant	118
5.6.1	Some $SU(3)$ -Gauge Theory	119
5.6.2	Definition of the Invariant	119
5.6.3	Properties and Computations	121
6	Instanton Floer Homology	123
6.1	Gauge Theory in Dimension 4	123
6.1.1	Gauge Theory on Closed 4-Manifolds	123
6.1.2	Gauge Theory on Open 4-Manifolds	125
6.1.3	Linear Analysis	126
6.1.4	Non-linear Analysis	129
6.2	Definition of the Floer Homology	130
6.2.1	Review of the Morse Theory	131
6.2.2	Floer Homology of Integral Homology Spheres	131
6.2.3	Functoriality with Respect to Cobordisms	135
6.3	Spectral Flow Formulas	137
6.3.1	The Atiyah-Patodi-Singer Formula	137
6.3.2	The Splitting Formula	139
6.3.3	The Kirk-Klassen Formula	141
6.4	Seifert Fibered and Graph Homology Spheres	143
6.4.1	The Algorithm	143
6.4.2	The Closed Form Formula	146
6.4.3	Graph Homology Spheres	148
6.5	Properties of the Floer Homology	149
6.5.1	Orientation Reversal	150
6.5.2	Floer Homology of Homology Handles	150
6.5.3	The Floer Exact Triangle	152

6.5.4	Special Boundary Maps	159
6.5.5	The u -map in Floer Homology	159
6.5.6	Integer Graded Floer Homology	161
6.5.7	Floer Homology of Connected Sums	163
6.5.8	Functoriality with Respect to Diffeomorphisms	166
6.5.9	Mutation Invariance	167
6.6	Floer Homology in Donaldson Theory	167
6.6.1	Donaldson Invariants of Closed 4-Manifolds	167
6.6.2	Relative Donaldson Polynomials	171
6.6.3	The Gluing Formula	175
6.7	Extending Floer Homology	177
6.7.1	Equivariant Floer Homology	177
6.7.2	Fukaya–Floer Homology	179
6.7.3	Floer’s Category and Functor	179
6.7.4	The Atiyah–Floer Conjecture	181
6.7.5	Floer Homology of Knots	181
6.7.6	Seiberg–Witten Floer Homology	183
7	The Homology Cobordism Group	185
7.1	Homology Cobordisms	185
7.1.1	Mazur Homology Spheres	185
7.1.2	Knot Cobordisms and Homology Cobordisms	187
7.1.3	Ribbon Concordances and Ribbon Knots	189
7.1.4	Branched Coverings	190
7.2	The Structure of $\Theta_{\mathbb{Z}}^3$	190
7.2.1	A Classical Approach to $\Theta_{\mathbb{Z}}^3$	191
7.2.2	Infinite Order Elements in $\Theta_{\mathbb{Z}}^3$	191
7.2.3	The $\bar{\mu}$ -Invariant	196
7.2.4	The Fukumoto-Furuta Invariants	197
7.2.5	The Group $\Theta_{\mathbb{Z}}^3$ is Infinitely Generated	199
7.3	Applications of the Homology Cobordism Group	200
7.3.1	Triangulating Topological Manifolds	200
7.3.2	Knot Concordance Group	200
7.3.3	PL-discs in Contractible 4-Manifolds	201
7.3.4	Constructing Smooth Manifolds	202
	References	205
	Index	219