## Contents

Introduction ..... 1
CHAPTER I
Basic Structural Tricks and Examples ..... 5

1. Algebras and their Defining Relations ..... 5
2. Skew Polynomial Rings ..... 9
3. $\mathbb{Z}$-filtrations and their Associated Graded Structures ..... 13
4. Homogenization and Dehomogenization of $\mathbb{Z}$-graded Rings ..... 25
5. Some Algebras: Classical and Modern ..... 28
CHAPTER II
Gröbner Bases in Associative Algebras ..... 33
6. (Left) Monomial Orderings and (Left) Admissible Systems ..... 34
7. (Left) Quasi-zero Elements and a (Left) Division Algorithm ..... 38
8. (Left) Gröbner Bases in a (Left) Admissible System ..... 42
9. (Left) Gröbner Bases in Various Contexts ..... 45
10. (Left) S-elements and Buchberger Theorem ..... 48
11. (Left) Dickson Systems and (Left) G-Noetherian Algebras ..... 54
12. Solvable Polynomial Algebras ..... 58
13. No (Left) Monomial Ordering Existing on $\Delta\left(k\left[x_{1}, \ldots, x_{n}\right]\right)$ with chark $>0$ ..... 61
CHAPTER III Gröbner Bases and Basic Algebraic-Algorithmic Structures ..... 67
14. PBW Bases of Finitely Generated Algebras ..... 68
15. Quadric Solvable Polynomial Algebras ..... 73
16. Associated Homogeneous Defining Relations of Algebras ..... 81
17. A Remark on Recognizable Properties of Algebras via Gröbner Bases ..... 89
CHAPTER IV
Filtered-Graded Transfer of Gröbner Bases ..... 91
18. Filtered-Graded Transfer of (Left) Admissible Systems ..... 92
19. Filtered-Graded Transfer of (Left) Gröbner Bases ..... 97
20. Filtered-Graded Transfer of (Left) Dickson Systems ..... 100
21. Filtered-Graded Transfer Applied to Quadric Solvable Polynomial Algebras ..... 103
CHAPTER V
GK-dimension of Modules over Quadric Solvable Polynomial Algebras and Elimination of Variables ..... 107
22. Gröbner Bases in Homogeneous Solvable Polynomial Algebras ..... 108
23. The Hilbert Function of $A / L$ ..... 110
24. The Hilbert Polynomial of $A / L$ ..... 112
25. GK-dimension Computation and Elimination of Variables (Homogeneous Case ) ..... 115
26. GK-dimension Computation and Elimination of Variables (Linear Case) ..... 119
27. The $\succeq_{g r}$-filtration on a Quadric Solvable Polynomial Algebra ..... 124
28. GK-dimension Computation and Elimination of Variables (General Quadric Case) ..... 126
29. Finite Dimensional Cyclic Modules ..... 130
CHAPTER VI
Multiplicity Computation of Modules over Quadric Solvable Polynomial Algebras ..... 133
30. The Multiplicity $e(M)$ of a Module $M$ ..... 134
31. Computation of $e(M)$ ..... 135
32. Computation of $\operatorname{GK} \cdot \operatorname{dim}\left(M \otimes_{k} N\right)$ and $e\left(M \otimes_{k} N\right)$ ..... 144
33. An Application to $A_{n}\left(q_{1}, \ldots, q_{n}\right)$ ..... 148
CHAPTER VII
$(\partial-)$ Holonomic Modules and Functions over Quadric Solvable Polynomial Algebras ..... 153
34. Some Operator Algebras ..... 154
35. Holonomic Functions ..... 156
36. Automatic Proving of Holonomic Function Identities ..... 162
37. Extension/Contraction of the $\partial$-finiteness ..... 164
5 . The $\partial$-holonomicity ..... 168
CHAPTER VIII
Regularity and $K_{0}$-group of Quadric Solvable Polynomial Algebras ..... 175
38. Tame Case: $A$ is Auslander Regular with $K_{0}(A) \cong \mathbb{Z}$ ..... 176
39. The $\succeq_{g r}$-filtration on Modules ..... 177
40. General Case: gl.dim $A \leq n$ ..... 181
41. General Case: $K_{0}(A) \cong \mathbb{Z}$ ..... 186
References ..... 187
Index ..... 195
