

Table of Contents

Algebraic Theory	1
1 Picard-Vessiot Theory	3
1.1 Differential Rings and Fields	3
1.2 Linear Differential Equations	6
1.3 Picard-Vessiot Extensions	12
1.4 The Differential Galois Group	18
1.5 Liouvillian Extensions	33
2 Differential Operators and Differential Modules	37
2.1 The Ring $\mathcal{D} = k[\partial]$ of Differential Operators	37
2.2 Constructions with Differential Modules	42
2.3 Constructions with Differential Operators	47
2.4 Differential Modules and Representations	53
3 Formal Local Theory	59
3.1 Formal Classification of Differential Equations	59
3.1.1 Regular Singular Equations	63
3.1.2 Irregular Singular Equations	68
3.2 The Universal Picard-Vessiot Ring of \widehat{K}	71
3.3 Newton Polygons	86
4 Algorithmic Considerations	99
4.1 Rational and Exponential Solutions	100
4.2 Factoring Linear Operators	110
4.2.1 Beke's Algorithm	111
4.2.2 Eigenring and Factorizations	114

4.3	Liouvillian Solutions	116
4.3.1	Group Theory	117
4.3.2	Liouvillian Solutions for a Differential Module	118
4.3.3	Liouvillian Solutions for a Differential Operator	121
4.3.4	Second Order Equations	125
4.3.5	Third Order Equations	128
4.4	Finite Differential Galois groups	130
4.4.1	Generalities on Scalar Fuchsian Equations	131
4.4.2	Restrictions on the Exponents	133
4.4.3	Representations of Finite Groups	134
4.4.4	A Calculation of the Accessory Parameter	135
4.4.5	Examples	136
	Analytic Theory	141
5	Monodromy, the Riemann-Hilbert Problem, and the Differential Galois Group	143
5.1	Monodromy of a Differential Equation	143
5.1.1	Local Theory of Regular Singular Equations	144
5.1.2	Regular Singular Equations on \mathbf{P}^1	147
5.2	A Solution of the Inverse Problem	150
5.3	The Riemann-Hilbert Problem	153
6	Differential Equations on the Complex Sphere and the Riemann-Hilbert Problem	157
6.1	Differentials and Connections	157
6.2	Vector Bundles and Connections	160
6.3	Fuchsian Equations	168
6.3.1	From Scalar Fuchsian to Matrix Fuchsian	168
6.3.2	A Criterion for a Scalar Fuchsian Equation	172
6.4	The Riemann-Hilbert Problem, Weak Form	174
6.5	Irreducible Connections	175
6.6	Counting Fuchsian Equations	181

7	Exact Asymptotics	187
	7.1 Introduction and Notation	187
	7.2 The Main Asymptotic Existence Theorem	193
	7.3 The Inhomogeneous Equation of Order One	199
	7.4 The Sheaves \mathcal{A} , \mathcal{A}^0 , $\mathcal{A}_{1/k}$, $\mathcal{A}_{1/k}^0$	203
	7.5 The Equation $(\delta - q)\hat{f} = g$ Revisited	208
	7.6 The Laplace and Borel Transforms	209
	7.7 The k -Summation Theorem	212
	7.8 The Multisummation Theorem	218
8	Stokes Phenomenon and Differential Galois Groups	229
	8.1 Introduction	229
	8.2 The Additive Stokes Phenomenon	230
	8.3 Construction of the Stokes Matrices	234
9	Stokes Matrices and Meromorphic Classification	245
	9.1 Introduction	245
	9.2 The Category Gr_2	246
	9.3 The Cohomology Set $H^1(S^1, STS)$	248
	9.4 Explicit 1-cocycles for $H^1(S^1, STS)$	252
	9.4.1 One Level k	254
	9.4.2 Two Levels $k_1 < k_2$	256
	9.4.3 The General Case	257
	9.5 $H^1(S^1, STS)$ as an Algebraic Variety	259
10	Universal Picard-Vessiot Rings and Galois Groups	261
	10.1 Introduction	261
	10.2 Regular Singular Differential Equations	262
	10.3 Formal Differential Equations	263
	10.4 Meromorphic Differential Equations	264
11	Inverse Problems	271
	11.1 Introduction	271
	11.2 The Inverse Problem for $\mathbf{C}((z))$	273

11.3	Some Topics on Linear Algebraic Groups	274
11.4	The Local Theorem	278
11.5	The Global Theorem	282
11.6	More on Abhyankar's Conjecture	284
11.7	The Constructive Inverse Problem	286
12	Moduli for Singular Differential Equations	295
12.1	Introduction	295
12.2	The Moduli Functor	297
12.3	An Example	299
12.3.1	Construction of the Moduli Space	299
12.3.2	Comparison with the Meromorphic Classification	300
12.3.3	Invariant Line Bundles	303
12.3.4	The Differential Galois Group	305
12.4	Unramified Irregular Singularities	306
12.5	The Ramified Case	310
12.6	The Meromorphic Classification	313
13	Positive Characteristic	317
13.1	Classification of Differential Modules	317
13.2	Algorithmic Aspects	322
13.2.1	The Equation $b^{(p-1)} + b^p = a$	323
13.2.2	The p -Curvature and Its Minimal Polynomial	324
13.2.3	Example: Operators of Order Two	326
13.3	Iterative Differential Modules	327
13.3.1	Picard-Vessiot Theory and Some Examples	328
13.3.2	Global Iterative Differential Equations	331
13.3.3	p -Adic Differential Equations	333
	Appendices	337
A	Algebraic Geometry	339
A.1	Affine Varieties	342
A.1.1	Basic Definitions and Results	342

A.1.2	Products of Affine Varieties over k	351
A.1.3	Dimension of an Affine Variety	355
A.1.4	Tangent Spaces, Smooth Points, and Singular Points	357
A.2	Linear Algebraic Groups	360
A.2.1	Basic Definitions and Results	360
A.2.2	The Lie Algebra of a Linear Algebraic Group	368
A.2.3	Torsors	370
B	Tannakian Categories	373
B.1	Galois Categories	373
B.2	Affine Group Schemes	377
B.3	Tannakian Categories	383
C	Sheaves and Cohomology	391
C.1	Sheaves: Definition and Examples	391
C.1.1	Germes and Stalks	393
C.1.2	Sheaves of Groups and Rings	394
C.1.3	From Presheaf to Sheaf	395
C.1.4	Moving Sheaves	397
C.1.5	Complexes and Exact Sequences	398
C.2	Cohomology of Sheaves	401
C.2.1	The Idea and the Formalism	401
C.2.2	Construction of the Cohomology Groups	404
C.2.3	More Results and Examples	408
D	Partial Differential Equations	409
D.1	The Ring of Partial Differential Operators	409
D.2	Picard-Vessiot Theory and Some Remarks	414
	Bibliography	419
	List of Notation	433
	Index	435