

Table of Contents

Introduction	1
1 Differentiable manifolds	3
1.1 Embedded manifolds in \mathbb{R}^N	4
1.2 The tangent space	7
1.3 The derivative of a differentiable function	8
1.4 Tangent and cotangent bundles of a manifold	9
1.5 Discontinuous action of a group on a manifold	9
1.6 Immersions and embeddings. Submanifolds	11
1.7 Partition of unity	12
2 Vector fields, differential forms and tensor fields	13
2.1 Lie derivative of tensor fields	16
2.2 The Henri Cartan formula	20
3 Pseudo-Riemannian manifolds	23
3.1 Affine connections	25
3.2 The Levi-Civita connection	29
3.3 Tubular neighborhood	34
3.4 Curvature	36
3.5 E. Cartan structural equations of a connection	50
4 Newtonian mechanics	55
4.1 Galilean space-time structure and Newton equations	55
4.2 Critical remarks on Newtonian mechanics	60
5 Mechanical systems on Riemannian manifolds	61
5.1 The generalized Newton law	61
5.2 The Jacobi Riemannian metric	63
5.3 Mechanical systems as second order vector fields	66
5.4 Mechanical systems with holonomic constraints	68
5.5 Some classical examples	70
5.6 The dynamics of rigid bodies	78
5.7 Dynamics of pseudo-rigid bodies	102
5.8 Dissipative mechanical systems	107

6	Mechanical systems with non-holonomic constraints	111
6.1	D'Alembert principle	111
6.2	Orientability of a distribution and conservation of volume . . .	119
6.3	Semi-holonomic constraints	123
6.4	The attractor of a dissipative system	123
7	Hyperbolicity and Anosov systems. Vakonomic mechanics	127
7.1	Hyperbolic and partially hyperbolic structures	127
7.2	Vakonomic mechanics	133
7.2.1	Some Hilbert manifolds	135
7.2.2	Lagrangian functionals and \mathcal{D} -spaces	136
7.3	D'Alembert versus vakonomics	136
7.4	Study of the \mathcal{D} -spaces	137
7.4.1	The tangent spaces of $H^1(M, \mathcal{D}, [a_0, a_1], m_0)$	137
7.4.2	The \mathcal{D} -space $H^1(M, \mathcal{D}, [a_0, a_1], m_0, m_1)$. Singular curves	140
7.5	Equations of motion in vakonomic mechanics	142
8	Special relativity	145
8.1	Lorentz manifolds	145
8.2	The quadratic map of \mathbb{R}_1^{n+1}	147
8.3	Time-cones and time-orientability of a Lorentz manifold	150
8.4	Lorentz geometry notions in special relativity	153
8.5	Minkowski space-time geometry	155
8.6	Lorentz and Poincaré groups	162
9	General relativity	165
9.1	Einstein equation	165
9.2	Geometric aspects of the Einstein equation	166
9.3	Schwarzschild space-time	169
9.4	Schwarzschild horizon	175
9.5	Light rays, Fermat principle and the deflection of light	175
A	Hamiltonian and Lagrangian formalisms	183
A.1	Hamiltonian systems	183
A.2	Euler–Lagrange equations	185
B	Möbius transformations and the Lorentz group	195
B.1	The Lorentz group	195
B.2	Stereographic projection	199
B.3	Complex structure of S^2	201
B.4	Möbius transformations	204
B.5	Möbius transformations and the proper Lorentz group	207
B.6	Lie algebra of the Lorentz group	211
B.7	Spinors	215
B.8	The sky of a rapidly moving observer	217

C	Quasi-Maxwell form of Einstein's equation	223
	C.1 Stationary regions, space manifold and global time	223
	C.2 Connection forms and equations of motion	226
	C.3 Stationary Maxwell equations	230
	C.4 Curvature forms and Ricci tensor	231
	C.5 Quasi-Maxwell equations	235
	C.6 Examples	239
D	Viscosity solutions and Aubry–Mather theory	245
	D.1 Optimal control and time independent problems	245
	D.2 Hamiltonian systems and the Hamilton–Jacobi theory	249
	D.3 Aubry–Mather theory	251
	References	259
	Index	263