

# Contents

Introduction .....	1
Chapter 1. Classical Calculus of Variations .....	5
§1. Euler Equation .....	5
1.1. Brachistochrone Problem .....	5
1.2. Euler Equation .....	6
1.3. Geodesics on a Riemannian Manifold .....	9
§2. Hamiltonian Formalism .....	12
2.1. Legendre Transform .....	12
2.2. Canonical Variables .....	15
2.3. Mechanical Meaning of the Canonical Variables .....	16
2.4. Variation Formula for a Functional with Movable Endpoints ..	17
2.5. Transversality Conditions in the Problem with Movable Endpoints .....	18
2.6. Weierstrass–Erdmann Conditions .....	20
2.7. Hamilton–Jacobi Equation .....	23
§3. Theory of the Second Variation .....	25
3.1. Problem of the Second Variation .....	25
3.2. Legendre Necessary Condition .....	26
3.3. The Associated Problem and the Definition of a Conjugate Point .....	28
3.4. Necessary Conditions for the Positive Semidefiniteness of $\delta^2 J$ ..	29
§4. Riccati Equation .....	30
4.1. Sufficient Conditions for the Positive Definiteness of $\delta^2 J$ ..	30
§5. Morse Index .....	35
§6. Jacobi Envelope Theorem .....	38
§7. Strong Minimum .....	42
7.1. Weierstrass Necessary Condition .....	42
§8. Poincaré–Cartan Integral Invariant .....	45
8.1. Exterior Differential Forms .....	45
8.2. Poincaré–Cartan Integral Invariant .....	47
8.3. Legendre Manifolds .....	50
§9. Fields of Extremals .....	53
9.1. Hilbert Invariant Integral .....	53
9.2. Embedding an Extremal in a Field and Focal Points .....	55
Chapter 2. Riccati Equation in the Classical Calculus of Variations ..	60
§1. Riccati Equation as a Sufficient Condition for Positivity of the Second Variation .....	60
§2. Riccati Equation for a Problem with Differential Constraints .....	62
2.1. Problem with Differential Constraints .....	63

2.2. Optimal Control Problem .....	63
2.3. Linear-Quadratic Problem .....	64
2.4. Bellman Equation .....	66
§3. Riccati Equation and the Grassmann Manifold .....	68
3.1. Grassmann Manifold.....	69
3.2. Riccati Equation as a Flow on the Grassmann Manifold .....	70
§4. Grassmann Manifolds of Lower Dimension .....	72
4.1. Quaternions .....	73
4.2. Homotopic Paths.....	77
 Chapter 3. Lie Groups and Lie Algebras .....	80
§1. Lie Groups: Definition and Examples .....	80
§2. Lie Algebras .....	84
2.1. Vector Fields on a Manifold .....	85
2.2. Lie Algebras .....	87
§3. Lie Groups of Lower Dimension .....	88
3.1. Topological Structure of the Groups $\text{SO}(3)$ and $\text{Spin}(3)$ .....	88
3.2. Topological Structure of the Group $\text{SL}(2, \mathbb{R})$ .....	90
3.3. Topological Structure of the Groups $\text{Sp}(1, \mathbb{R})$ , $\text{U}(1)$ , and $\text{SU}(2)$ .....	90
§4. Adjoint Representation and Killing Form .....	91
4.1. Adjoint Representation .....	91
4.2. Killing Form .....	93
4.3. Subalgebras and Ideals.....	93
§5. Semisimple Lie Groups .....	96
5.1. Compact Lie Algebras .....	98
§6. Homogeneous and Symmetrical Spaces .....	100
6.1. Symmetrical Spaces .....	101
§7. Totally Geodesic Submanifolds .....	107
7.1. Lie Group Isometries .....	107
7.2. Geodesics in the Quotient Space of Lie Groups .....	110
 Chapter 4. Grassmann Manifolds .....	112
§1. Three Approaches to the Description of the Grassmann Manifolds ..	112
1.1. Local Coordinates on the Grassmann Manifold .....	112
1.2. Invariant Description of Grassmann Manifolds .....	114
1.3. Metric on the Grassmann Manifold .....	114
1.4. Grassmann Manifolds as Symmetrical Spaces .....	114
1.5. Plücker Embeddings .....	115
§2. Lagrange–Grassmann Manifolds .....	117
2.1. Coordinates on a Lagrange–Grassmann Manifold .....	118
2.2. Lagrange–Grassmann Manifold as a Homogeneous Space .....	119
2.3. The Manifold $\Lambda(\mathbb{R}^{2n})$ as a Symmetrical Space.....	123
§3. Riccati Equation as a Flow on the Manifold $G_n(\mathbb{R}^{2n})$ .....	123

§4. Systems Associated with a Linear System of Differential Equations . . . . .	126
4.1. Associated Systems on Grassmann Manifolds . . . . .	128
Chapter 5. Matrix Double Ratio . . . . .	130
§1. Matrix Double Ratio on the Grassmann Manifold . . . . .	130
§2. Clifford Algebras . . . . .	135
§3. Totally Geodesic Submanifolds of Grassmann Manifolds . . . . .	137
§4. Curves with a Scalar Double Ratio . . . . .	143
§5. Fourth Harmonic as a Geodesic Symmetry . . . . .	147
5.1. Manifold of Isotropic Planes . . . . .	148
§6. Clifford Parallels . . . . .	150
§7. Connection Between Clifford Parallels and Isoclinic Planes . . . . .	154
§8. Matrix Double Ratio on the Lagrange–Grassmann Manifold . . . . .	155
§9. Morse–Maslov–Arnol'd Index in the Leray–Kashivara Form . . . . .	158
§10. Fourth Harmonic as an Isometry of the Lagrange–Grassmann Manifold . . . . .	162
§11. Application of the Matrix Double Ratio to the Study of the Riccati Equation . . . . .	162
Chapter 6. Complex Riccati Equations . . . . .	166
§1. Cartan–Siegel Domains . . . . .	166
§2. Klein–Poincaré Upper Half-Plane and Generalized Siegel Upper Half-Plane . . . . .	174
2.1. Generalized Siegel Upper Half-Plane . . . . .	177
2.2. Siegel Half-Plane as a Symmetrical Space . . . . .	178
2.3. Action of $\mathrm{Sp}(n, \mathbb{R})$ on the Boundary of the Siegel Half-Plane . . . . .	184
2.4. Cayley Transform . . . . .	185
§3. Complexified Riccati Equation as a Flow on the Generalized Siegel Upper Half-Plane . . . . .	187
§4. Flow on Cartan–Siegel Homogeneity Domains . . . . .	189
4.1. Riccati-Type Equation for a Linear System Whose Matrix Belongs to a Given Lie Algebra . . . . .	190
4.2. Flow on the Siegel Homogeneity Domain of Type I . . . . .	192
4.3. Flow on the Siegel Homogeneity Domain of Type II . . . . .	194
4.4. Flow on the Siegel Homogeneity Domain of Type IV . . . . .	196
§5. Matrix Analog of the Schwarz Differential Operator . . . . .	198
5.1. Classical Schwarz Differential Operator . . . . .	200
5.2. Schwarz Operator and a Linear Second-Order Differential Equation . . . . .	202
5.3. Schwarz Operator and the Riccati Equation . . . . .	203
5.4. Matrix Analog of the Schwarz Operator . . . . .	205

Chapter 7. Higher-Dimensional Calculus of Variations . . . . .	208
§1. Minimal Surfaces . . . . .	208
§2. Necessary Optimality Conditions for a Multiple Integral . . . . .	212
2.1. Euler Equation . . . . .	213
2.2. Second Variation . . . . .	215
2.3. Variational Equation . . . . .	216
§3. Vector Bundles . . . . .	217
§4. Distributions and the Frobenius Theorem . . . . .	219
§5. Connection in a Linear Bundle . . . . .	227
§6. Levi-Civita Connection . . . . .	230
6.1. Torsion and Curvature of a Connection of a Vector Bundle . . . . .	233
§7. Nonnegativity Conditions of the Second Variation . . . . .	236
§8. Field Theory in the Weyl Form . . . . .	240
§9. Caratheodory Transformation . . . . .	244
9.1. Condition for Realizability of the Caratheodory Transformation . . . . .	247
§10. Field Theory in the Caratheodory Form . . . . .	248
Chapter 8. On the Quadratic System of Partial Differential Equations Related to the Minimization Problem for a Multiple Integral . . . . .	254
§1. Riccati Equation in the Case of the Degenerate Legendre Condition . . . . .	254
§2. Reducing the Dirichlet Integral to the Integral of Its Principal Part . . . . .	257
§3. Relation of the Riccati Partial Differential Equation to the Euler Equation . . . . .	261
3.1. Compactification of the Space on Which the Riccati Partial Differential Equation is Defined . . . . .	263
§4. Connection Defined by a Solution to the Riccati Partial Differential Equation . . . . .	264
4.1. Potentiability Condition for Tensor Fields . . . . .	270
Epilogue . . . . .	272
Appendix to the English Edition . . . . .	273
References . . . . .	276
Index . . . . .	282