
Contents

Notation	XI
1 Basics	1
1.1 Classification and Correctness	1
1.2 Fourier's Method, Integral Transforms	5
1.3 Maximum Principle, Fundamental Solution	9
1.3.1 Elliptic Boundary Value Problems	9
1.3.2 Parabolic Equations and Initial-Boundary Value Problems	15
1.3.3 Hyperbolic Initial and Initial-Boundary Value Problems	18
2 Finite Difference Methods	23
2.1 Basic Concepts	23
2.2 Illustrative Examples	31
2.3 Transportation Problems and Conservation Laws	36
2.3.1 The One-Dimensional Linear Case	37
2.3.2 Properties of Nonlinear Conservation Laws	48
2.3.3 Difference Methods for Nonlinear Conservation Laws...	53
2.4 Elliptic Boundary Value Problems	61
2.4.1 Elliptic Boundary Value Problems	61
2.4.2 The Classical Approach to Finite Difference Methods ..	62
2.4.3 Discrete Green's Function	74
2.4.4 Difference Stencils and Discretization in General Domains	76
2.4.5 Mixed Derivatives, Fourth Order Operators	82
2.4.6 Local Grid Refinements	89
2.5 Finite Volume Methods as Finite Difference Schemes	90
2.6 Parabolic Initial-Boundary Value Problems	103
2.6.1 Problems in One Space Dimension	104
2.6.2 Problems in Higher Space Dimensions	109
2.6.3 Semi-Discretization	113

2.7	Second-Order Hyperbolic Problems	118
3	Weak Solutions	125
3.1	Introduction	125
3.2	Adapted Function Spaces	128
3.3	Variational Equations and Conforming Approximation	142
3.4	Weakening V-ellipticity	163
3.5	Nonlinear Problems	167
4	The Finite Element Method	173
4.1	A First Example	173
4.2	Finite-Element-Spaces	178
4.2.1	Local and Global Properties	178
4.2.2	Examples of Finite Element Spaces in \mathbb{R}^2 and \mathbb{R}^3	189
4.3	Practical Aspects of the Finite Element Method	202
4.3.1	Structure of a Finite Element Code	202
4.3.2	Description of the Problem	203
4.3.3	Generation of the Discrete Problem	205
4.3.4	Mesh Generation and Manipulation	210
4.4	Convergence of Conforming Methods	217
4.4.1	Interpolation and Projection Error in Sobolev Spaces	217
4.4.2	Hilbert Space Error Estimates	227
4.4.3	Inverse Inequalities and Pointwise Error Estimates	232
4.5	Nonconforming Finite Element Methods	238
4.5.1	Introduction	238
4.5.2	Ansatz Spaces with Low Smoothness	239
4.5.3	Numerical Integration	244
4.5.4	The Finite Volume Method Analysed from a Finite Element Viewpoint	251
4.5.5	Remarks on Curved Boundaries	254
4.6	Mixed Finite Elements	258
4.6.1	Mixed Variational Equations and Saddle Points	258
4.6.2	Conforming Approximation of Mixed Variational Equations	265
4.6.3	Weaker Regularity for the Poisson and Biharmonic Equations	272
4.6.4	Penalty Methods and Modified Lagrange Functions	277
4.7	Error Estimators and Adaptive FEM	287
4.7.1	The Residual Error Estimator	288
4.7.2	Averaging and Goal-Oriented Estimators	292
4.8	The Discontinuous Galerkin Method	294
4.8.1	The Primal Formulation for a Reaction-Diffusion Problem	295
4.8.2	First-Order Hyperbolic Problems	299
4.8.3	Error Estimates for a Convection-Diffusion Problem	302

4.9	Further Aspects of the Finite Element Method	306
4.9.1	Conditioning of the Stiffness Matrix	306
4.9.2	Eigenvalue Problems	307
4.9.3	Superconvergence	310
4.9.4	p- and hp-Versions	314
5	Finite Element Methods for Unsteady Problems	317
5.1	Parabolic Problems	317
5.1.1	On the Weak Formulation	317
5.1.2	Semi-Discretization by Finite Elements	321
5.1.3	Temporal Discretization by Standard Methods	330
5.1.4	Temporal Discretization with Discontinuous Galerkin Methods	337
5.1.5	Rothe's Method	343
5.1.6	Error Control	347
5.2	Second-Order Hyperbolic Problems	356
5.2.1	Weak Formulation of the Problem	356
5.2.2	Semi-Discretization by Finite Elements	358
5.2.3	Temporal Discretization	363
5.2.4	Rothe's Method for Hyperbolic Problems	368
5.2.5	Remarks on Error Control	372
6	Singularly Perturbed Boundary Value Problems	375
6.1	Two-Point Boundary Value Problems	376
6.1.1	Analytical Behaviour of the Solution	376
6.1.2	Discretization on Standard Meshes	383
6.1.3	Layer-adapted Meshes	394
6.2	Parabolic Problems, One-dimensional in Space	399
6.2.1	The Analytical Behaviour of the Solution	399
6.2.2	Discretization	401
6.3	Convection-Diffusion Problems in Several Dimensions	406
6.3.1	Analysis of Elliptic Convection-Diffusion Problems	406
6.3.2	Discretization on Standard Meshes	412
6.3.3	Layer-adapted Meshes	427
6.3.4	Parabolic Problems, Higher-Dimensional in Space	430
7	Variational Inequalities, Optimal Control	435
7.1	Analytic Properties	435
7.2	Discretization of Variational Inequalities	447
7.3	Penalty Methods	457
7.3.1	Basic Concept of Penalty Methods	457
7.3.2	Adjustment of Penalty and Discretization Parameters	473
7.4	Optimal Control of PDEs	480
7.4.1	Analysis of an Elliptic Model Problem	480
7.4.2	Discretization by Finite Element Methods	489

8	Numerical Methods for Discretized Problems	499
8.1	Some Particular Properties of the Problems	499
8.2	Direct Methods	502
8.2.1	Gaussian Elimination for Banded Matrices	502
8.2.2	Fast Solution of Discrete Poisson Equations, FFT	504
8.3	Classical Iterative Methods	510
8.3.1	Basic Structure and Convergence	510
8.3.2	Jacobi and Gauss-Seidel Methods	514
8.3.3	Block Iterative Methods	520
8.3.4	Relaxation and Splitting Methods	524
8.4	The Conjugate Gradient Method	530
8.4.1	The Basic Idea, Convergence Properties	530
8.4.2	Preconditioned CG Methods	538
8.5	Multigrid Methods	548
8.6	Domain Decomposition, Parallel Algorithms	560
	Bibliography: Textbooks and Monographs	571
	Bibliography: Original Papers	577
	Index	585