

---

# Contents

---

## Part I Foundations

---

<b>1</b>	<b>Propositional Logic</b> .....	3
1.1	Syntax .....	4
1.2	Semantics .....	6
1.3	Satisfiability and Validity .....	8
1.3.1	Truth Tables .....	9
1.3.2	Semantic Arguments .....	10
1.4	Equivalence and Implication .....	14
1.5	Substitution .....	16
1.6	Normal Forms .....	18
1.7	Decision Procedures for Satisfiability .....	21
1.7.1	Simple Decision Procedures .....	21
1.7.2	Reconsidering the Truth-Table Method .....	22
1.7.3	Conversion to an Equisatisfiable Formula in CNF .....	24
1.7.4	The Resolution Procedure .....	27
1.7.5	DPLL .....	28
1.8	Summary .....	31
	Bibliographic Remarks .....	32
	Exercises .....	32
<b>2</b>	<b>First-Order Logic</b> .....	35
2.1	Syntax .....	35
2.2	Semantics .....	39
2.3	Satisfiability and Validity .....	42
2.4	Substitution .....	45
2.4.1	Safe Substitution .....	47
2.4.2	Schema Substitution .....	48
2.5	Normal Forms .....	51
2.6	Decidability and Complexity .....	53
2.6.1	Satisfiability as a Formal Language .....	53

2.6.2	Decidability .....	54
2.6.3	★Complexity .....	54
2.7	★Meta-Theorems of First-Order Logic .....	56
2.7.1	Simplifying the Language of FOL .....	57
2.7.2	Semantic Argument Proof Rules .....	58
2.7.3	Soundness and Completeness .....	58
2.7.4	Additional Theorems .....	61
2.8	Summary .....	66
	Bibliographic Remarks .....	67
	Exercises .....	67
<b>3</b>	<b>First-Order Theories</b> .....	<b>69</b>
3.1	First-Order Theories .....	69
3.2	Equality .....	71
3.3	Natural Numbers and Integers .....	73
3.3.1	Peano Arithmetic .....	73
3.3.2	Presburger Arithmetic .....	75
3.3.3	Theory of Integers .....	76
3.4	Rationals and Reals .....	79
3.4.1	Theory of Reals .....	80
3.4.2	Theory of Rationals .....	82
3.5	Recursive Data Structures .....	84
3.6	Arrays .....	87
3.7	★Survey of Decidability and Complexity .....	90
3.8	Combination Theories .....	91
3.9	Summary .....	92
	Bibliographic Remarks .....	93
	Exercises .....	93
<b>4</b>	<b>Induction</b> .....	<b>95</b>
4.1	Stepwise Induction .....	95
4.2	Complete Induction .....	99
4.3	Well-Founded Induction .....	102
4.4	Structural Induction .....	108
4.5	Summary .....	110
	Bibliographic Remarks .....	111
	Exercises .....	111
<b>5</b>	<b>Program Correctness: Mechanics</b> .....	<b>113</b>
5.1	pi: A Simple Imperative Language .....	114
5.1.1	The Language .....	115
5.1.2	Program Annotations .....	118
5.2	Partial Correctness .....	123
5.2.1	Basic Paths: Loops .....	125
5.2.2	Basic Paths: Function Calls .....	131

5.2.3	Program States .....	135
5.2.4	Verification Conditions .....	136
5.2.5	$P$ -Invariant and $P$ -Inductive .....	142
5.3	Total Correctness .....	143
5.4	Summary .....	149
	Bibliographic Remarks .....	150
	Exercises .....	151
<b>6</b>	<b>Program Correctness: Strategies</b> .....	<b>153</b>
6.1	Developing Inductive Annotations .....	153
6.1.1	Basic Facts .....	154
6.1.2	The Precondition Method .....	156
6.1.3	A Strategy .....	162
6.2	Extended Example: QuickSort .....	164
6.2.1	Partial Correctness .....	167
6.2.2	Total Correctness .....	171
6.3	Summary .....	172
	Bibliographic Remarks .....	173
	Exercises .....	173

---

**Part II Algorithmic Reasoning**

---

<b>7</b>	<b>Quantified Linear Arithmetic</b> .....	<b>183</b>
7.1	Quantifier Elimination .....	184
7.1.1	Quantifier Elimination .....	184
7.1.2	A Simplification .....	185
7.2	Quantifier Elimination over Integers .....	185
7.2.1	Augmented Theory of Integers .....	185
7.2.2	Cooper's Method .....	187
7.2.3	A Symmetric Elimination .....	194
7.2.4	Eliminating Blocks of Quantifiers .....	195
7.2.5	★Solving Divides Constraints .....	196
7.3	Quantifier Elimination over Rationals .....	200
7.3.1	Ferrante and Rackoff's Method .....	200
7.4	★Complexity .....	204
7.5	Summary .....	204
	Bibliographic Remarks .....	205
	Exercises .....	205
<b>8</b>	<b>Quantifier-Free Linear Arithmetic</b> .....	<b>207</b>
8.1	Decision Procedures for Quantifier-Free Fragments .....	207
8.2	Preliminary Concepts and Notation .....	209
8.3	Linear Programs .....	213
8.4	The Simplex Method .....	218

8.4.1	From $M$ to $M_0$ . . . . .	219
8.4.2	Vertex Traversal . . . . .	223
8.4.3	★Complexity . . . . .	237
8.5	Summary . . . . .	237
	Bibliographic Remarks . . . . .	238
	Exercises . . . . .	238
<b>9</b>	<b>Quantifier-Free Equality and Data Structures</b> . . . . .	<b>241</b>
9.1	Theory of Equality . . . . .	242
9.2	Congruence Closure Algorithm . . . . .	244
9.2.1	Relations . . . . .	245
9.2.2	Congruence Closure Algorithm . . . . .	247
9.3	Congruence Closure with DAGs . . . . .	251
9.3.1	Directed Acyclic Graphs . . . . .	251
9.3.2	Basic Operations . . . . .	254
9.3.3	Congruence Closure Algorithm . . . . .	255
9.3.4	Decision Procedure for $T_E$ -Satisfiability . . . . .	256
9.3.5	★Complexity . . . . .	258
9.4	Recursive Data Structures . . . . .	259
9.5	Arrays . . . . .	263
9.6	Summary . . . . .	265
	Bibliographic Remarks . . . . .	266
	Exercises . . . . .	267
<b>10</b>	<b>Combining Decision Procedures</b> . . . . .	<b>269</b>
10.1	Combining Decision Procedures . . . . .	269
10.2	Nelson-Oppen Method: Nondeterministic Version . . . . .	271
10.2.1	Phase 1: Variable Abstraction . . . . .	271
10.2.2	Phase 2: Guess and Check . . . . .	273
10.2.3	Practical Efficiency . . . . .	274
10.3	Nelson-Oppen Method: Deterministic Version . . . . .	276
10.3.1	Convex Theories . . . . .	276
10.3.2	Phase 2: Equality Propagation . . . . .	278
10.3.3	Equality Propagation: Implementation . . . . .	282
10.4	★Correctness of the Nelson-Oppen Method . . . . .	283
10.5	★Complexity . . . . .	287
10.6	Summary . . . . .	288
	Bibliographic Remarks . . . . .	288
	Exercises . . . . .	288
<b>11</b>	<b>Arrays</b> . . . . .	<b>291</b>
11.1	Arrays with Uninterpreted Indices . . . . .	292
11.1.1	Array Property Fragment . . . . .	292
11.1.2	Decision Procedure . . . . .	294
11.2	Integer-Indexed Arrays . . . . .	299

11.2.1	Array Property Fragment .....	300
11.2.2	Decision Procedure .....	301
11.3	Hashtables .....	304
11.3.1	Hashtable Property Fragment .....	305
11.3.2	Decision Procedure .....	306
11.4	Larger Fragments .....	308
11.5	Summary .....	309
	Bibliographic Remarks .....	310
	Exercises .....	310
<b>12</b>	<b>Invariant Generation .....</b>	<b>311</b>
12.1	Invariant Generation .....	311
12.1.1	Weakest Precondition and Strongest Postcondition .....	312
12.1.2	★General Definitions of wp and sp .....	315
12.1.3	Static Analysis .....	316
12.1.4	Abstraction .....	319
12.2	Interval Analysis .....	325
12.3	Karr's Analysis .....	333
12.4	★Standard Notation and Concepts .....	341
12.5	Summary .....	344
	Bibliographic Remarks .....	345
	Exercises .....	345
<b>13</b>	<b>Further Reading .....</b>	<b>347</b>
	<b>References .....</b>	<b>351</b>
	<b>Index .....</b>	<b>357</b>