
Contents

Part I Introduction and Basic Concepts

1	Introduction and Overview	3
1.1	Motivation and Background	6
1.1.1	Quillen Complexes	6
1.1.2	Minimal Free Resolutions of Certain Semigroup Algebras	7
1.1.3	Lie Algebras	8
1.1.4	Disconnected k -hypergraphs and Subspace Arrangements	9
1.1.5	Cohomology of Spaces of Knots	10
1.1.6	Determinantal Ideals	11
1.1.7	Other Examples	12
1.1.8	Links to Graph Theory	13
1.1.9	Complexity Theory and Evasiveness	14
1.2	Overview	14
2	Abstract Graphs and Set Systems	19
2.1	Graphs, Hypergraphs, and Digraphs	19
2.1.1	Graphs	20
2.1.2	Paths, Components and Cycles	20
2.1.3	Bipartite Graphs	21
2.1.4	Digraphs	21
2.1.5	Directed Paths and Cycles	21
2.1.6	Hypergraphs	22
2.1.7	General Terminology	22
2.2	Posets and Lattices	23
2.3	Abstract Simplicial Complexes	23
2.3.1	Basic Definitions	24
2.3.2	Dimension	24
2.3.3	Collapses	24
2.3.4	Joins, Cones, Suspensions, and Wedges	25
2.3.5	Alexander Duals	25

2.3.6	Links and Deletions	25
2.3.7	Lifted Complexes	25
2.3.8	Order Complexes and Face Posets	25
2.3.9	Graph, Digraph, and Hypergraph Complexes and Properties	26
2.4	Matroids	26
2.4.1	Graphic Matroids	27
2.5	Integer Partitions	28
3	Simplicial Topology	29
3.1	Simplicial Homology	29
3.2	Relative Homology	31
3.3	Homotopy Theory	32
3.4	Contractible Complexes and Their Relatives	35
3.4.1	Acyclic and k -acyclic Complexes	35
3.4.2	Contractible and k -connected Complexes	36
3.4.3	Collapsible Complexes	37
3.4.4	Nonevasive Complexes	38
3.5	Quotient Complexes	38
3.6	Shellable Complexes and Their Relatives	40
3.6.1	Cohen-Macaulay Complexes	40
3.6.2	Constructible Complexes	41
3.6.3	Shellable Complexes	41
3.6.4	Vertex-Decomposable Complexes	42
3.6.5	Topological Properties and Relations Between Different Classes	43
3.7	Balls and Spheres	46
3.8	Stanley-Reisner Rings	47

Part II Tools

4	Discrete Morse Theory	51
4.1	Informal Discussion	51
4.2	Acyclic Matchings	53
4.3	Simplicial Morse Theory	55
4.4	Discrete Morse Theory on Complexes of Groups	59
4.4.1	Independent Sets in the Homology of a Complex	61
4.4.2	Simple Applications	64
5	Decision Trees	67
5.1	Basic Properties of Decision Trees	69
5.1.1	Element-Decision Trees	69
5.1.2	Set-Decision Trees	70
5.2	Hierarchy of Almost Nonevasive Complexes	72

5.2.1	Semi-nonevasive and Semi-collapsible Complexes	73
5.2.2	Relations Between Some Important Classes of Complexes	76
5.3	Some Useful Constructions	79
5.4	Further Properties of Almost Nonevasive Complexes	81
5.5	A Potential Generalization	86
6	Miscellaneous Results	87
6.1	Posets	87
6.2	Depth	88
6.3	Vertex-Decomposability	92
6.4	Enumeration	93

Part III Overview of Graph Complexes

7	Graph Properties	99
7.1	List of Complexes	100
7.2	Illustrations	104
8	Dihedral Graph Properties	107
8.1	Basic Definitions	108
8.2	List of Complexes	109
8.3	Illustrations	111
9	Digraph Properties	113
9.1	List of Complexes	113
9.2	Illustrations	117
10	Main Goals and Proof Techniques	119
10.1	Homology	119
10.2	Homotopy Type	120
10.3	Connectivity Degree	120
10.4	Depth	120
10.5	Euler Characteristic	123
10.6	Remarks on Nonevasiveness and Related Properties	123

Part IV Vertex Degree

11	Matchings	127
11.1	Some General Results	128
11.2	Complete Graphs	130
11.2.1	Rational Homology	130
11.2.2	Homotopical Depth and Bottom Nonvanishing Homology	131

XII Contents

11.2.3 Torsion in Higher-Degree Homology Groups	136
11.2.4 Further Properties	141
11.3 Chessboards	143
11.3.1 Bottom Nonvanishing Homology	143
11.3.2 Torsion in Higher-Degree Homology Groups	145
11.4 Paths and Cycles	148
12 Graphs of Bounded Degree	151
12.1 Bounded-Degree Graphs Without Loops	152
12.1.1 The Case $d = 2$	153
12.1.2 The General Case	155
12.2 Bounded-Degree Graphs with Loops	161
12.3 Euler Characteristic	165

Part V Cycles and Crossings

13 Forests and Matroids	171
13.1 Independence Complexes	172
13.2 Pseudo-Independence Complexes	173
13.2.1 PI Graph Complexes	175
13.3 Strong Pseudo-Independence Complexes	176
13.3.1 Sets in Matroids Avoiding Odd Cycles	181
13.3.2 SPI Graph Complexes	182
13.4 Alexander Duals of SPI Complexes	184
13.4.1 SPI* Monotone Graph Properties	186
14 Bipartite Graphs	189
14.1 Bipartite Graphs Without Restrictions	190
14.2 Disconnected Bipartite Graphs	192
14.3 Unbalanced Bipartite Graphs	193
14.3.1 Depth	194
14.3.2 Homotopy Type	195
14.3.3 Euler Characteristic	198
14.3.4 Generalization to Hypergraphs	203
15 Directed Variants of Forests and Bipartite Graphs	205
15.1 Directed Forests	206
15.2 Acyclic Digraphs	206
15.3 Bipartite Digraphs	208
15.4 Graded Digraphs	209
15.5 Digraphs with No Non-alternating Circuits	213
15.6 Digraphs Without Odd Directed Cycles	213

16 Noncrossing Graphs 217

 16.1 The Associahedron 218

 16.2 A Shelling of the Associahedron 219

 16.3 Noncrossing Matchings 222

 16.4 Noncrossing Forests 226

 16.5 Noncrossing Bipartite Graphs 229

17 Non-Hamiltonian Graphs 233

 17.1 Homotopy Type 234

 17.2 Homology 240

 17.3 Directed Variant 242

Part VI Connectivity

18 Disconnected Graphs 245

 18.1 Disconnected Graphs Without Restrictions 246

 18.2 Graphs with No Large Components 247

 18.2.1 Homotopy Type and Depth 248

 18.2.2 Bottom Nonvanishing Homology Group 253

 18.3 Graphs with Some Small Components 258

 18.4 Graphs with Some Component of Size Not Divisible by p 262

 18.5 Disconnected Hypergraphs 262

19 Not 2-connected Graphs 263

 19.1 Homotopy Type 263

 19.2 Homology 268

 19.3 A Decision Tree 269

 19.4 Generalization and Yet Another Proof 271

20 Not 3-connected Graphs and Beyond 275

 20.1 Homotopy Type 275

 20.2 Homology 283

 20.3 A Related Polytopal Sphere 287

 20.4 Not k -connected Graphs for $k > 3$ 289

21 Dihedral Variants of k -connected Graphs 291

 21.1 A General Observation 292

 21.2 Graphs with a Disconnected Polygon Representation 292

 21.3 Graphs with a Separable Polygon Representation 294

 21.4 Graphs with a Two-separable Polygon Representation 298

22 Directed Variants of Connected Graphs 301

 22.1 Not Strongly Connected Digraphs 301

 22.2 Not Strongly 2-connected Digraphs 306

 22.3 Non-spanning Digraphs 307

23 Not 2-edge-connected Graphs	309
23.1 An Acyclic Matching	310
23.2 Enumerative Properties of the Given Matching	318
23.3 Bottom Nonvanishing Homology Group	321
23.4 Top Nonvanishing Homology Group	323

Part VII Cliques and Stable Sets

24 Graphs Avoiding k-matchings	329
25 t-colorable Graphs	333
26 Graphs and Hypergraphs with Bounded Covering Number	337
26.1 Solid Hypergraphs	338
26.2 A Related Simplicial Complex	340
26.3 An Acyclic Matching	341
26.4 Homotopy Type and Homology	343
26.5 Computations	347
26.6 Homotopical Depth	350
26.7 Triangle-Free Graphs	351
26.8 Concluding Remarks and Open Problems	352

Part VIII Open Problems

27 Open Problems	357
References	363
Index	371