

Contents

Preface	ix
1 Eigenvalues of elliptic operators	1
1.1 Notation and prerequisites	1
1.1.1 Notation and Sobolev spaces	1
1.1.2 Partial differential equations	2
1.2 Eigenvalues and eigenfunctions	4
1.2.1 Abstract spectral theory	4
1.2.2 Application to elliptic operators	5
1.2.3 First Properties of eigenvalues	8
1.2.4 Regularity of eigenfunctions	9
1.2.5 Some examples	9
1.2.6 Fredholm alternative	11
1.3 Min-max principles and applications	12
1.3.1 Min-max principles	12
1.3.2 Monotonicity	13
1.3.3 Nodal domains	14
1.4 Perforated domains	15
2 Tools	17
2.1 Schwarz rearrangement	17
2.2 Steiner symmetrization	18
2.2.1 Definition	18
2.2.2 Properties	20
2.2.3 Continuous Steiner symmetrization	21
2.3 Continuity of eigenvalues	23
2.3.1 Introduction	23
2.3.2 Continuity with variable coefficients	26
2.3.3 Continuity with variable domains (Dirichlet case)	28
2.3.4 The case of Neumann eigenvalues	33
2.4 Two general existence theorems	35
2.5 Derivatives of eigenvalues	37

2.5.1	Introduction	37
2.5.2	Derivative with respect to the domain	38
2.5.3	Case of multiple eigenvalues	41
2.5.4	Derivative with respect to coefficients	43
3	The first eigenvalue of the Laplacian-Dirichlet	45
3.1	Introduction	45
3.2	The Faber-Krahn inequality	45
3.3	The case of polygons	46
3.3.1	An existence result	47
3.3.2	The cases $N = 3, 4$	50
3.3.3	A challenging open problem	51
3.4	Domains in a box	52
3.5	Multi-connected domains	55
4	The second eigenvalue of the Laplacian-Dirichlet	61
4.1	Minimizing λ_2	61
4.1.1	The Theorem of Krahn-Szegö	61
4.1.2	Case of a connectedness constraint	63
4.2	A convexity constraint	63
4.2.1	Optimality conditions	64
4.2.2	Geometric properties of the optimal domain	67
4.2.3	Another regularity result	71
5	The other Dirichlet eigenvalues	73
5.1	Introduction	73
5.2	Connectedness of minimizers	74
5.3	Existence of a minimizer for λ_3	76
5.3.1	A concentration-compactness result	76
5.3.2	Existence of a minimizer	77
5.4	Case of higher eigenvalues	80
6	Functions of Dirichlet eigenvalues	85
6.1	Introduction	85
6.2	Ratio of eigenvalues	86
6.2.1	The Ashbaugh-Benguria Theorem	86
6.2.2	Some other ratios	90
6.2.3	A collection of open problems	92
6.3	Sums of eigenvalues	93
6.3.1	Sums of eigenvalues	93
6.3.2	Sums of inverses	94
6.4	General functions of λ_1 and λ_2	95
6.4.1	Description of the set $\mathcal{E} = (\lambda_1, \lambda_2)$	95
6.4.2	Existence of minimizers	98

7	Other boundary conditions for the Laplacian	101
7.1	Neumann boundary condition	101
7.1.1	Introduction	101
7.1.2	Maximization of the second Neumann eigenvalue	102
7.1.3	Some other problems	104
7.2	Robin boundary condition	106
7.2.1	Introduction	106
7.2.2	The Bossel-Daners Theorem	107
7.2.3	Optimal insulation of conductors	110
7.3	Stekloff eigenvalue problem	113
8	Eigenvalues of Schrödinger operators	117
8.1	Introduction	117
8.1.1	Notation	117
8.1.2	A general existence result	119
8.2	Maximization or minimization of the first eigenvalue	119
8.2.1	Introduction	119
8.2.2	The maximization problem	119
8.2.3	The minimization problem	123
8.3	Maximization or minimization of other eigenvalues	125
8.4	Maximization or minimization of the fundamental gap $\lambda_2 - \lambda_1$	127
8.4.1	Introduction	127
8.4.2	Single-well potentials	127
8.4.3	Minimization or maximization with an L^∞ constraint	131
8.4.4	Minimization or maximization with an L^p constraint	134
8.5	Maximization of ratios	136
8.5.1	Introduction	136
8.5.2	Maximization of $\lambda_2(V)/\lambda_1(V)$ in one dimension	136
8.5.3	Maximization of $\lambda_n(V)/\lambda_1(V)$ in one dimension	137
9	Non-homogeneous strings and membranes	141
9.1	Introduction	141
9.2	Existence results	143
9.2.1	A first general existence result	143
9.2.2	A more precise existence result	143
9.2.3	Nonlinear constraint	146
9.3	Minimizing or maximizing $\lambda_k(\rho)$ in dimension 1	148
9.3.1	Minimizing $\lambda_k(\rho)$	149
9.3.2	Maximizing $\lambda_k(\rho)$	150
9.4	Minimizing or maximizing $\lambda_k(\rho)$ in higher dimension	152
9.4.1	Case of a ball	152
9.4.2	General case	153
9.4.3	Some extensions	155

10 Optimal conductivity	159
10.1 Introduction	159
10.2 The one-dimensional case	160
10.2.1 A general existence result	160
10.2.2 Minimization or maximization of $\lambda_k(\sigma)$	161
10.2.3 Case of Neumann boundary conditions	163
10.3 The general case	165
10.3.1 The maximization problem	165
10.3.2 The minimization problem	168
11 The bi-Laplacian operator	169
11.1 Introduction	169
11.2 The clamped plate	169
11.2.1 History	169
11.2.2 Notation and statement of the theorem	170
11.2.3 Proof of the Rayleigh conjecture in dimension $N = 2, 3$	171
11.3 Buckling of a plate	174
11.3.1 Introduction	174
11.3.2 The case of a positive eigenfunction	175
11.3.3 An existence result	177
11.3.4 The last step in the proof	178
11.4 Some other problems	181
11.4.1 Non-homogeneous rod and plate	181
11.4.2 The optimal shape of a column	183
References	187
Index	199