

# Contents

Preface . . . . .	ix
<b>1 Observability and Controllability for Finite-dimensional Systems</b>	
1.1 Norms and inner products . . . . .	1
1.2 Operators on finite-dimensional spaces . . . . .	5
1.3 Matrix exponentials . . . . .	7
1.4 Observability and controllability for finite-dimensional linear systems . . . . .	11
1.5 The Hautus test and Gramians . . . . .	15
<b>2 Operator Semigroups</b>	
2.1 Strongly continuous semigroups and their generators . . . . .	20
2.2 The spectrum and the resolvents of an operator . . . . .	24
2.3 The resolvents of a semigroup generator and the space $\mathcal{D}(A^\infty)$ . . . . .	28
2.4 Invariant subspaces for semigroups . . . . .	33
2.5 Riesz bases . . . . .	36
2.6 Diagonalizable operators and semigroups . . . . .	40
2.7 Strongly continuous groups . . . . .	47
2.8 The adjoint semigroup . . . . .	53
2.9 The embeddings $V \subset H \subset V'$ . . . . .	56
2.10 The spaces $X_1$ and $X_{-1}$ . . . . .	59
2.11 Bounded perturbations of a generator . . . . .	65
<b>3 Semigroups of Contractions</b>	
3.1 Dissipative and m-dissipative operators . . . . .	69
3.2 Self-adjoint operators . . . . .	73
3.3 Positive operators . . . . .	78
3.4 The spaces $H_{\frac{1}{2}}$ and $H_{-\frac{1}{2}}$ . . . . .	81
3.5 Sturm–Liouville operators . . . . .	88
3.6 The Dirichlet Laplacian . . . . .	92
3.7 Skew-adjoint operators . . . . .	98

3.8	The theorems of Lumer–Phillips and Stone . . . . .	102
3.9	The wave equation with boundary damping . . . . .	106
<b>4</b>	<b>Control and Observation Operators</b>	
4.1	Solutions of non-homogeneous equations . . . . .	112
4.2	Admissible control operators . . . . .	116
4.3	Admissible observation operators . . . . .	121
4.4	The duality between the admissibility concepts . . . . .	126
4.5	Two representation theorems . . . . .	128
4.6	Infinite-time admissibility . . . . .	134
4.7	Remarks and bibliographical notes on Chapter 4 . . . . .	136
<b>5</b>	<b>Testing Admissibility</b>	
5.1	Gramians and Lyapunov inequalities . . . . .	139
5.2	Admissible control operators for left-invertible semigroups . . . . .	144
5.3	Admissibility for diagonal semigroups . . . . .	147
5.4	Some unbounded perturbations of generators . . . . .	157
5.5	Admissible control operators for perturbed semigroups . . . . .	164
5.6	Remarks and bibliographical notes on Chapter 5 . . . . .	168
<b>6</b>	<b>Observability</b>	
6.1	Some observability concepts . . . . .	173
6.2	Some examples based on the string equation . . . . .	179
6.3	Robustness of exact observability with respect to admissible perturbations of the generator . . . . .	184
6.4	Simultaneous exact observability . . . . .	190
6.5	A Hautus-type necessary condition for exact observability . . . . .	194
6.6	Hautus-type tests for exact observability with a skew-adjoint generator . . . . .	197
6.7	From $\ddot{w} = -A_0 w$ to $\dot{z} = iA_0 z$ . . . . .	200
6.8	From first- to second-order equations . . . . .	205
6.9	Spectral conditions for exact observability with a skew-adjoint generator . . . . .	211
6.10	The clamped Euler–Bernoulli beam with torque observation at an endpoint . . . . .	217
6.11	Remarks and bibliographical notes on Chapter 6 . . . . .	220
<b>7</b>	<b>Observation for the Wave Equation</b>	
7.1	An admissibility result for boundary observation . . . . .	226
7.2	Boundary exact observability . . . . .	231
7.3	A perturbed wave equation . . . . .	234
7.4	The wave equation with distributed observation . . . . .	240
7.5	Some consequences for the Schrödinger and plate equations . . . . .	247

7.6	The wave equation with boundary damping and boundary velocity observation . . . . .	251
7.7	Remarks and bibliographical notes on Chapter 7 . . . . .	257
<b>8</b>	<b>Non-harmonic Fourier Series and Exact Observability</b>	
8.1	A theorem of Ingham . . . . .	261
8.2	Variable coefficients PDEs in one space dimension with boundary observation . . . . .	266
8.3	Domains associated with a sequence . . . . .	270
8.4	The results of Kahane and Beurling . . . . .	276
8.5	The Schrödinger and plate equations in a rectangular domain with distributed observation . . . . .	280
8.6	Remarks and bibliographical notes on Chapter 8 . . . . .	284
<b>9</b>	<b>Observability for Parabolic Equations</b>	
9.1	Preliminary results . . . . .	287
9.2	From $\ddot{w} = -A_0 w$ to $\dot{z} = -A_0 z$ . . . . .	289
9.3	Final state observability with geometric conditions . . . . .	295
9.4	A global Carleman estimate for the heat operator . . . . .	298
9.5	Final state observability without geometric conditions . . . . .	312
9.6	Remarks and bibliographical notes on Chapter 9 . . . . .	314
<b>10</b>	<b>Boundary Control Systems</b>	
10.1	What is a boundary control system? . . . . .	317
10.2	Two simple examples in one space dimension . . . . .	322
10.2.1	A one-dimensional heat equation with Neumann boundary control . . . . .	323
10.2.2	A string equation with Neumann boundary control . . . . .	324
10.3	A string equation with variable coefficients . . . . .	326
10.4	An Euler–Bernoulli beam with torque control . . . . .	330
10.5	An Euler–Bernoulli beam with angular velocity control . . . . .	334
10.6	The Dirichlet map on an $n$ -dimensional domain . . . . .	337
10.7	Heat and Schrödinger equations with boundary control . . . . .	341
10.8	The convection-diffusion equation with boundary control . . . . .	344
10.9	The wave equation with Dirichlet boundary control . . . . .	347
10.10	Remarks and bibliographical notes on Chapter 10 . . . . .	352
<b>11</b>	<b>Controllability</b>	
11.1	Some controllability concepts . . . . .	355
11.2	The duality between controllability and observability . . . . .	357
11.3	Simultaneous controllability and the reachable space with $\mathcal{H}^1$ inputs . . . . .	364
11.4	An example of a coupled system . . . . .	371

11.5	Null-controllability for heat and convection-diffusion equations . . . . .	375
11.6	Boundary controllability for Schrödinger and wave equations . . . . .	378
11.6.1	Boundary controllability for the Schrödinger equation . . . . .	379
11.6.2	Boundary controllability for the wave equation . . . . .	380
11.7	Remarks and bibliographical notes on Chapter 11 . . . . .	381
<b>12</b>	<b>Appendix I: Some Background on Functional Analysis</b>	
12.1	The closed-graph theorem and some consequences . . . . .	385
12.2	Compact operators . . . . .	387
12.3	The square root of a positive operator . . . . .	391
12.4	The Fourier and Laplace transformations . . . . .	394
12.5	Banach space-valued $L^p$ functions . . . . .	399
<b>13</b>	<b>Appendix II: Some Background on Sobolev Spaces</b>	
13.1	Test functions . . . . .	404
13.2	Distributions on a domain . . . . .	408
13.3	The operators div, grad, rot and $\Delta$ . . . . .	412
13.4	Definition and first properties of Sobolev spaces . . . . .	415
13.5	Regularity of the boundary and Sobolev spaces on manifolds . . . . .	420
13.6	Trace operators and the space $\mathcal{H}_{\Gamma_0}^1(\Omega)$ . . . . .	424
13.7	The Green formulas and extensions of trace operators . . . . .	430
<b>14</b>	<b>Appendix III: Some Background on Differential Calculus</b>	
14.1	Critical points and Sard's theorem . . . . .	435
14.2	Existence of Morse functions on $\Omega$ . . . . .	437
14.3	Proof of Theorem 9.4.3 . . . . .	441
<b>15</b>	<b>Appendix IV: Unique Continuation for Elliptic Operators</b>	
15.1	A Carleman estimate for elliptic operators . . . . .	445
15.2	The unique continuation results . . . . .	454
<b>Bibliography</b>		459
<b>List of Notation</b>		475
<b>Index</b>		477